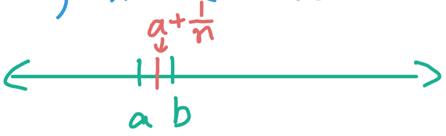
Lecture 4: Highlights

MAJOR RESULT #1

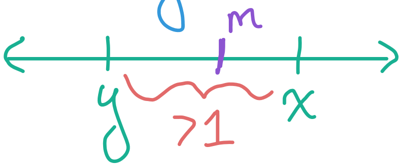
Thm (Archimedean Property): If $a, b \in \mathbb{R}$ satisfy $a > 0$ and $b > 0$, then there exists $n \in \mathbb{N}$ so that $n\underset{\uparrow \text{ spoon}}{a} > \underset{\uparrow \text{ bathtub}}{b}$.

Lemma: For any $a \in \mathbb{R}$, there exists $n \in \mathbb{N}$ s.t. $a < n$.

Lemma: For any $a, b \in \mathbb{R}$, $a < b$, there exists $n \in \mathbb{N}$ so that $a + \frac{1}{n} < b$.



Lemma: If $x, y \in \mathbb{R}$ satisfy $1 < x - y$, then $\exists\, m \in \mathbb{Z}$ so that $y < m < x$.



MAJOR THEOREM #2

Thm ($\mathbb{Q}$ is dense in $\mathbb{R}$): If $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ satisfying $a < r < b$.

Prop: For $a, b \in \mathbb{R}$, $a < b$, the set $S = [a, b)$ does not have a maximum and $\sup(S) = b$.

Def (Unbounded above/below): Suppose $S \subseteq \mathbb{R}$ is nonempty.

- If S is not bounded above, write $\sup(S) = +\infty$.
- If S is not bounded below, write $\inf(S) = -\infty$.

Remark: Given a nonempty set $S \subseteq \mathbb{R}$,

- By defn of supremum and $\mathbb{R}$

$$\begin{aligned} S \text{ has a supremum} &\iff S \text{ is bounded above} \\ &\iff \sup(S) \in \mathbb{R} \end{aligned}$$

- Similarly,

$$\begin{aligned} S \text{ doesn't have a supremum} &\iff S \text{ is not bounded above} \\ &\iff \sup(S) = +\infty \end{aligned}$$

Using this notation, even though not every set has a supremum, for any nonempty $S \subseteq \mathbb{R}$, $\sup(S)$ has meaning.