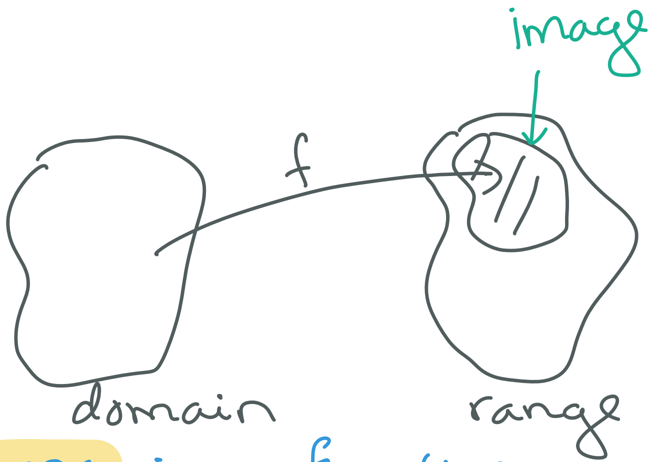
Lecture 5

Ch 2: Sequences

Recall: functions



Def (sequence): A sequence is a function whose domain is a set of the form $\{m, m+1, m+2, \ldots\}$ for some $m \in \mathbb{Z}$. We will study sequences whose range is $\mathbb{R}$.

Typically, the domain of a sequence will be either $\{0, 1, 2, 3, \ldots\}$ or $\{1, 2, 3, \ldots\}$

Remark:

To emphacize that a sequence is special type of function...

instead of writing $f(n)$, we write s_n

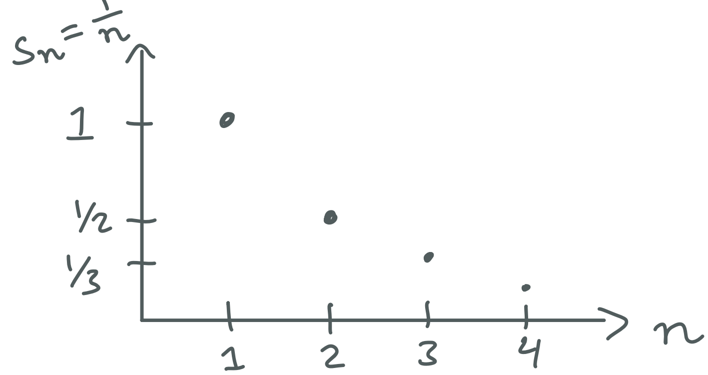
We'll often specify a sequence by listing its values in order, $(s_1, s_2, s_3, \ldots)$.

Ex: • If $s_n = \frac{1}{n}$ for $n \geq 1$, the sequence is $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$

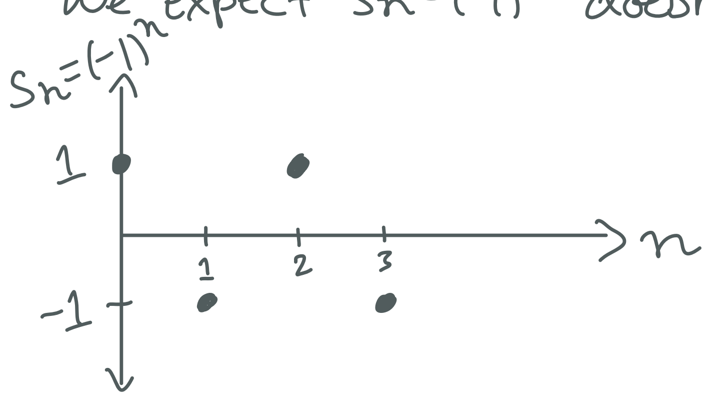
• If $s_n = (-1)^n$ for $n \geq 0$, the sequence is $(1, -1, 1, -1, \dots)$

Heuristically, a sequence "converges" to some limit $s \in \mathbb{R}$ if the values of s_n stay close to s for large n.

Ex: We expect $s_n = \frac{1}{n}$ converges to 0.



We expect $s_n = (-1)^n$ doesn't converge.



Def (convergence):

- A sequence s_n of real numbers **converges** to some $s \in \mathbb{R}$ provided that

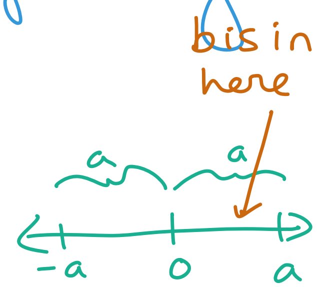
  [for all $\varepsilon > 0$, there exists $N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - s| < \varepsilon$.]

- The number s is the **limit** of s_n, and we write $\lim_{n\to\infty} s_n = s$ or $s_n \to s$.

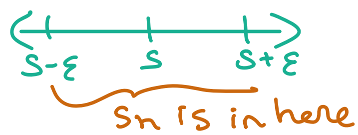
- A sequence that does not converge to any $s \in \mathbb{R}$ it is said to **diverge**.

Remark:

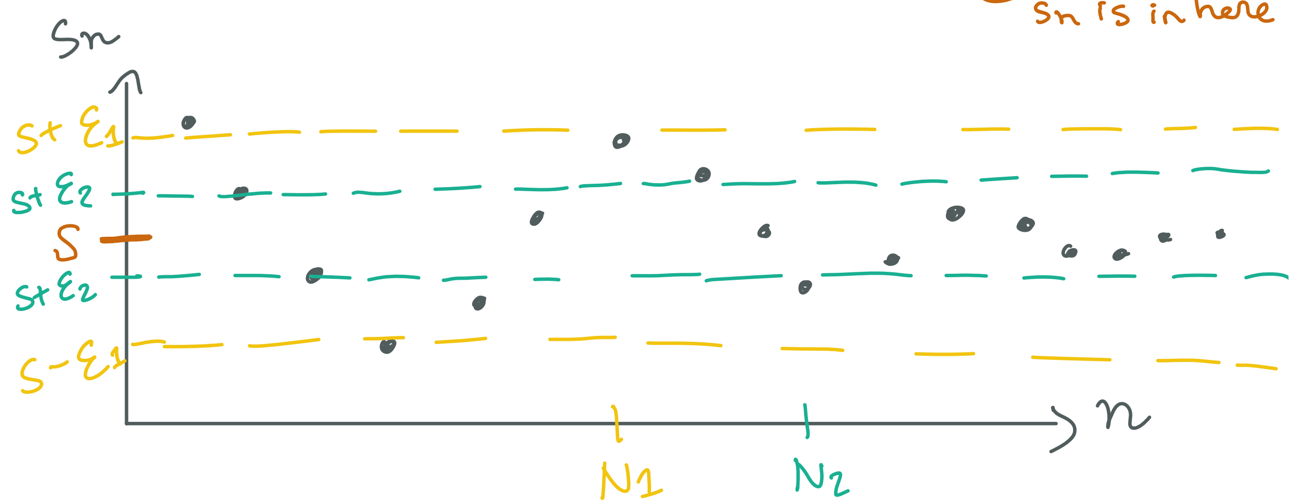
- Recall: $|b| < a \iff -a < b < a$



- Thus $|s_n - s| < \varepsilon \iff -\varepsilon < s_n - s < \varepsilon \iff s - \varepsilon < s_n < s + \varepsilon$



- N can depend on ε.



Assume domain of sequence is $\mathbb{N} = \{1, 2, 3, \ldots\}$, unless otherwise specified.

Ex: Consider the sequence $s_n = \frac{1}{n^2}$. We expect that $\lim_{n\to\infty} \frac{1}{n^2} = 0$. Let's prove this!

Scratchwork: $\left|\frac{1}{n^2} - 0\right| < \varepsilon \iff \frac{1}{n^2} < \varepsilon \iff \frac{1}{n} < \sqrt{\varepsilon} \iff \frac{1}{\sqrt{\varepsilon}} < n$

Proof: Fix arbitrary $\varepsilon > 0$. Let $N = \frac{1}{\sqrt{\varepsilon}}$. Then for $n > N$, we have

$$n > \frac{1}{\sqrt{\varepsilon}} \iff \frac{1}{n^2} < \varepsilon \iff \left|\frac{1}{n^2} - 0\right| < \varepsilon.$$ Thus $\lim_{n\to\infty} \frac{1}{n^2} = 0$. $\square$

Remark: We could have picked N to be any number $\geq \frac{1}{\sqrt{\varepsilon}}$, e.g. $N = \frac{2}{\sqrt{\varepsilon}}$, $N = \frac{1}{\sqrt{\varepsilon}} + \pi$, ...

Ex: Consider the sequence $s_n = (-1)^n$. We expect that this sequence does not converge. Let's prove it.

Proof:

Assume, for the sake of contradiction, that $(-1)^n$ converges to $s \in \mathbb{R}$. By defn of convergence, for all $\varepsilon > 0$, there exists N so that $n > N$, $|(-1)^n - s| < \varepsilon$.

Let $\varepsilon = 1$ and choose N so that $n > N$ ensures $|(-1)^n - s| < 1 \iff s-1 < (-1)^n < s+1$.

For n even, this implies $1 < s+1 \implies 0 < s$.
For n odd, this implies $s-1 < -1 \implies s < 0$.
This is a contradiction. Thus, $(-1)^n$ diverges. $\square$

Ex: Consider the sequence $s_n = \dfrac{2n-1}{3n+2}$.
What is the limit?

$\boxed{\text{Scratchwork:}}$

$$s_n = \frac{2n-1}{3n+2} = \frac{2 - \frac{1}{n}}{3 + \frac{2}{n}}$$

"these get very small as $n \to +\infty$"

$$\left|s_n - \frac{2}{3}\right| < \varepsilon \iff \left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| < \varepsilon \iff \left|\frac{6n-3-6n-4}{3(3n+2)}\right| < \varepsilon$$

$$\iff \left|\frac{-7}{3(3n+2)}\right| < \varepsilon \iff \frac{7}{3(3n+2)} < \varepsilon$$

$$\Leftarrow \frac{7}{9n} < \varepsilon \quad \Leftarrow \frac{1}{n} < \varepsilon \iff \frac{1}{\varepsilon} < n$$

Proof:

Fix $\varepsilon > 0$ arbitrary and let $N = \frac{1}{\varepsilon}$. Then, if $n > N$, we have

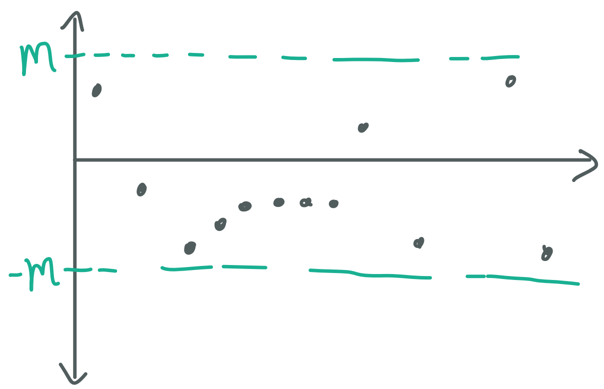
$$\frac{1}{\varepsilon} < n \Rightarrow \frac{7}{3(3n+2)} < \varepsilon \Leftrightarrow \left|\frac{6n-3-6n-4}{3(3n+2)}\right| < \varepsilon \Leftrightarrow \left|s_n - \frac{2}{3}\right| < \varepsilon.$$

Therefore, $\lim_{n \to \infty} s_n = \frac{2}{3}$. $\square$

A special type of sequence is a...

Def (bounded sequence): A sequence s_n is **bounded** if there exists $M \in \mathbb{R}$ s.t. $|s_n| \le M$ for all n.

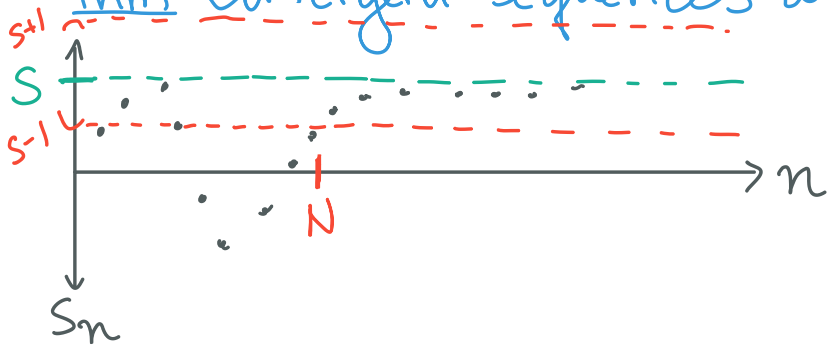
$\iff -M \le s_n \le M$



Remark: A sequence is bounded iff the set $S = \{s_n : n \in \mathbb{N}\}$ is bounded. (HW3)

$s_n = (-1)^n$, $S = \{-1, 1\}$, $s_n = \frac{1}{n^2}$, $S = \{1, \frac{1}{4}, \frac{1}{9}, \dots\}$

Thm: Convergent sequences are bounded.



Pf: Suppose s_n is a convergent sequence with limit s. Then, for $\varepsilon = 1$, there exists

WLOG, we may assume $N \ge 0$

$N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - s| < 1$

$\iff -1 < s_n - s < 1 \iff s - 1 < s_n < s + 1$. Let

Reverse Δ ineq.

$M_0 = \max\{|s+1|, |s-1|\}$. Then $|s_n| - |s| \le \big||s_n| - |s|\big| \le |s_n - s| < 1$

$\Rightarrow |s_n| < 1 + |s|$

$$-M_0 \le -|s-1| \le s-1 < s_n < s+1 \le |s+1| \le M_0$$

Thus, if $n > N$, we have $|s_n| \le M_0$.

$\max\{|s_n| : 1 \le n \le N, n \in \mathbb{N}\}$

Let $M_1 = \max\{|s_1|, |s_2|, |s_3|, \dots, |s_{\lfloor N \rfloor}|\}$.

$\lfloor N \rfloor = \max\{m \in \mathbb{Z} : m \le N\}$

Then, for $1 \le n \le N$, $|s_n| \le M_1$.

Finally, let $M = \max\{M_0, M_1\}$. Then $|s_n| \le M \ \forall n \in \mathbb{N}$, so our sequence is bounded. $\square$

<u>Remark</u>: The converse is not true, since <u>not</u> all bounded sequences are convergent, e.g., $(-1)^n$.