Lecture 6

Now, we will prove several limit theorems that will help us find the limits of more complicated sequences by breaking them into parts.

Thm (limit of sum is sum of limits): If s_n and t_n are convergent sequences, $\lim_{n\to\infty}(s_n+t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$.

Ex: $\lim_{n\to\infty}\left(\frac{\pi}{n} + \frac{\sqrt{2}}{n^2}\right) = \lim_{n\to\infty}\frac{\pi}{n} + \lim_{n\to\infty}\frac{\sqrt{2}}{n^2} = 0 + 0 = 0$

Recall: triangle inequality $|a+b| \leq |a| + |b|$.

Pf: Let $s = \lim_{n\to\infty} s_n$ and $t = \lim_{n\to\infty} t_n$. Fix $\varepsilon > 0$. We must show there exists $N \in \mathbb{R}$ so that $n > N$ ensures $|(s_n+t_n) - (s+t)| < \varepsilon$.

Note that $|(s_n+t_n) - (s+t)| \leq |s_n - s| + |t_n - t|$.

Since $s_n \to s$ and $t_n \to t$ given $\tilde{\varepsilon} = \frac{\varepsilon}{2} > 0$, there exists N_s and $N_t \in \mathbb{R}$ so that $n > N_s$ ensures $|s_n - s| < \tilde{\varepsilon}$ and $n > N_t$ ensures $|t_n - t| < \tilde{\varepsilon}$.
Let $N = \max\{N_s, N_t\}$. Then for all $n > N$.
$|(s_n + t_n) - (s+t)| \leq |s_n - s| + |t_n - t| < \tilde{\varepsilon} + \tilde{\varepsilon} = \varepsilon$. $\square$

Remark: The requirement that s_n and t_n are convergent sequences is necessary.
For example, $s_n = (-1)^n$, $t_n = (-1)^{n+1}$.

Then $\lim_{n\to\infty} s_n + t_n = 0$, but $\lim_{n\to\infty} s_n$ and $\lim_{n\to\infty} t_n$ do not exist.

Thm (limit of product is product of limits): If s_n and t_n are convergent sequences, $\lim_{n\to\infty} s_n t_n = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right)$

Exercise

Give an example to show that the assumption that s_n and t_n are convergent sequences is necessary for the previous theorem to be true.

Pf: Let $s = \lim_{n\to\infty} s_n$, $t = \lim_{n\to\infty} t_n$. Fix $\varepsilon > 0$.
We must show there exists $N \in \mathbb{R}$ so that $n > N$ ensures $|s_n t_n - st| < \varepsilon$.

Note that

"add and subtract"

$$
\begin{aligned}
|s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\
&\leq |s_n t_n - s_n t| + |s_n t - st| \\
&= |s_n||t_n - t| + |t||s_n - s|
\end{aligned}
$$

Since s_n is a convergent sequence, it is a bounded sequence, that is there exists M_s so that $|s_n| \leq M_s$ for all n. Define $M = \max\{M_s, |t|, 1\}$. ← ensures $M \geq M_s, M \geq |t|, M > 0$

Combining with estimates above, $|s_n t_n - st| \le M|t_n - t| + M|s_n - s|$.
For $\tilde{\varepsilon} = \frac{\varepsilon}{2M} > 0$, there exists N_s and N_t so that $n > N_s$ ensures $|s_n - s| < \tilde{\varepsilon}$ and $n > N_t$ ensures $|t_n - t| < \tilde{\varepsilon}$. Let $N = \max\{N_s, N_t\}$. Then for all $n > N$, $|s_n t_n - st| < M\tilde{\varepsilon} + M\tilde{\varepsilon} = \varepsilon$. $\square$

<u>Thm</u> (limit of quotient is quotient of limits): If s_n and t_n are convergent sequences, $s_n \neq 0$ for all n, and $\lim_{n\to\infty} s_n \neq 0$, then
$$\lim_{n\to\infty}\left(\frac{t_n}{s_n}\right) = \frac{\lim_{n\to\infty} t_n}{\lim_{n\to\infty} s_n}.$$

<u>Pf</u>: See textbook.

Thm (basic examples):

(a) $\lim_{n\to\infty} \left(\frac{1}{n}\right)^p = 0$ if $p>0$

(b) $\lim_{n\to\infty} a^n = 0$ if $|a|<1$

(c) $\lim_{n\to\infty} n^{1/n} = 1$

(d) $\lim_{n\to\infty} a^{1/n} = 1$ if $a>0$

Pf: See textbook.