Lecture 6: Highlights

Thm (limit of sum is sum of limits): If s_n and t_n are convergent sequences, $\lim_{n\to\infty}(s_n+t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$.

Thm (limit of product is product of limits): If s_n and t_n are convergent sequences, $\lim_{n\to\infty} s_n t_n = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right)$

Thm (limit of quotient is quotient of limits): If s_n and t_n are convergent sequences, $s_n \neq 0$ for all n, and $\lim_{n\to\infty} s_n \neq 0$, then

$$\lim_{n\to\infty}\left(\frac{t_n}{s_n}\right) = \frac{\lim_{n\to\infty} t_n}{\lim_{n\to\infty} s_n}.$$

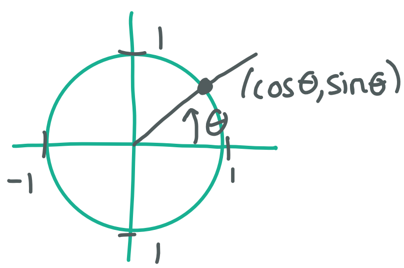
Thm (basic examples):

(a) $\lim_{n\to\infty} \left(\frac{1}{n}\right)^p = 0$ if $p>0$

(b) $\lim_{n\to\infty} a^n = 0$ if $|a|<1$

(c) $\lim_{n\to\infty} n^{1/n} = 1$

(d) $\lim_{n\to\infty} a^{1/n} = 1$ if $a>0$



Reminder: $\cos(n\pi) = (\underset{n=1}{-1}, \underset{n=2}{1}, -1, 1, -1, -1, \dots) = (-1)^n$

$\sin(n\pi) = (0, 0, 0, 0, \dots)$

Ex: Find the limit of $s_n = \frac{n-2}{n^2+2}$ and justify your answer.

Note that $s_n = \dfrac{\frac{1}{n} - \frac{2}{n^2}}{1 + \frac{2}{n^2}}$. Let $a_n = \frac{1}{n} - \frac{2}{n^2}$. Let $b_n = 1 + \frac{2}{n^2}$.

Since $(-2,-2,-2,\ldots)$ converges to -2 and, by Basic Examples Thm, $\frac{1}{n^2}$ converges to 0, by Thm that limit of product is product of limits, we have

$$\lim_{n\to\infty} \frac{-2}{n^2} = \left(\lim_{n\to\infty} -2\right)\left(\lim_{n\to\infty} \frac{1}{n^2}\right) = (-2)\cdot 0 = 0.$$

Furthermore, by Basic Ex Thm, $\frac{1}{n} \to 0$. By Thm that limit of sum is sum of limits, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} + \left(\frac{-2}{n^2}\right) = \left(\lim_{n\to\infty} \frac{1}{n}\right) + \left(\lim_{n\to\infty} \frac{-2}{n^2}\right) = 0 + 0 = 0.$$

Since we just showed $\lim_{n\to\infty} \frac{-2}{n^2} = 0$ and $\lim_{n\to\infty} -1 = -1$, by Thm that limit of product is product of limits,

$$\lim_{n\to\infty} \frac{2}{n^2} = \lim_{n\to\infty} \left(\frac{-2}{n^2}\right)(-1) = \left(\lim_{n\to\infty} \frac{-2}{n^2}\right)\left(\lim_{n\to\infty} -1\right) = 0\cdot(-1) = 0.$$

Finally, since $\lim_{n\to\infty} 1 = 1$, by Thm that limit of sum is sum of limits,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1 + \frac{2}{n^2} = \left(\lim_{n\to\infty} 1\right) + \left(\lim_{n\to\infty} \frac{2}{n^2}\right) = 1 + 0 = 1.$$

Since a_n converges, b_n converges, $b_n \neq 0 \ \forall n$, and $\lim_{n\to\infty} b_n \neq 0$, by Thm that limit of quotient is quotient of limit, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a_n}{b_n} = \dfrac{\left(\lim_{n\to\infty} a_n\right)}{\left(\lim_{n\to\infty} b_n\right)} = \frac{0}{1} = 0.$

Comments on HW1:

- A good number of students (maybe a quarter) seem to have misunderstandings using quantifiers like for all vs. there exists.

- Many of the write ups I looked at seemed to be closer to scratchwork than a complete proof. For example, in proving an induction statement they would write what they set out to show and then cross terms out, etc. until they arrived at something like the induction hypothesis. I assume that the quality of writing is as important here as figuring out the puzzle.

- Many students skipped steps or gave handwavy arguments, so it might be worth reiterating that they really need to use the axioms/lemmas/propositions/theorems rather than intuition.

Let $P(n)$ be the statement that $n! > n^2$ for $n \geq 4$.

<u>Base Case:</u> Consider the case of $n=4$, and since

$$4! = 24 > 16 = 4^2$$

$P(4)$ is true.

← use full sentences!

<u>Induction:</u> Assume $P(n)$ is true for some integer $n \geq 4$. Then,

$$n! > n^2 \quad \text{by the induction hypothesis}$$

$$(n+1)n! > (n+1)n^2 \quad \text{since } n+1 > 0 \text{ \& } n! > n^2$$

$$\vdots$$

$$(n+1)! > (n+1)^2$$

<u>Scratch:</u>

$(n+1)! > (n+1)^2$

$(n+1)n!$