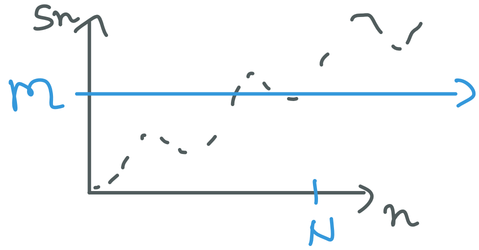
Lecture 7

Ex: What is the limit of $s_n = n^2$?

<u>Def</u> (diverges to $+\infty$ or $-\infty$): A sequence s_n diverges to $+\infty$ if for all $M>0$ there exists $N \in \mathbb{R}$ so that $n>N$ ensures $s_n > M$. We write $\lim_{n\to\infty} s_n = +\infty$.

Likewise, a sequence s_n diverges to $-\infty$ if for all $M<0$ there exists N so that $n>N$ ensures $s_n < M$.
We write $\lim_{n\to\infty} s_n = -\infty$.



Remark:
- If s_n diverges to $\pm\infty$, it does not converge.
- We will say that s_n "has a limit" or "the limit of s_n exists" if either
  1. s_n converges $\qquad \lim_{n\to\infty} s_n \in \mathbb{R}$
  2. s_n diverges to $\pm\infty$ $\qquad \lim_{n\to\infty} s_n \in \{+\infty, -\infty\}$

A few limit theorems for sequences that diverge to $+\infty$ or $-\infty$...

Case 1: $\lim_{n\to\infty} t_n = t,\ t>0$
Case 2: $\lim_{n\to\infty} t_n = +\infty$

Thm: Suppose $\lim_{n\to\infty} s_n = +\infty$ and $\lim_{n\to\infty} t_n > 0$. Then, $\lim_{n\to\infty} s_n t_n = +\infty$.

Pf: HW.

Thm: Suppose s_n is a sequence of positive real numbers. Then $\lim_{n\to\infty} s_n = +\infty \iff \lim_{n\to\infty} \frac{1}{s_n} = 0$.

Pf: First, suppose $\lim_{n\to\infty} s_n = +\infty$. Fix $\varepsilon > 0$. Note that $\left|\frac{1}{s_n} - 0\right| < \varepsilon \iff \frac{1}{s_n} < \varepsilon \iff \frac{1}{\varepsilon} < s_n$. Since s_n diverges to $+\infty$, there exists N s.t. $n > N$ ensures $s_n > \frac{1}{\varepsilon} \iff \left|\frac{1}{s_n} - 0\right| < \varepsilon$. Thus, $\lim_{n\to\infty} \frac{1}{s_n} = 0$.

Next, suppose $\lim_{n\to\infty} \frac{1}{s_n} = 0$. Fix $M > 0$. Note that $s_n > M \iff \frac{1}{s_n} < \frac{1}{M} \iff \left|\frac{1}{s_n} - 0\right| < \frac{1}{M}$. Since $\frac{1}{s_n}$ converges to 0, there exists N s.t. $n > N$ ensures $\left|\frac{1}{s_n} - 0\right| < \frac{1}{M} \iff s_n > M$. Thus, $\lim_{n\to\infty} s_n = +\infty$.

Thm: If $\lim_{n\to\infty} s_n = +\infty$, then $\lim_{n\to\infty}(-s_n) = -\infty$.

Pf: Fix $M<0$. Note that $(-s_n)<M \iff s_n > -M$. Since s_n diverges to $+\infty$, there exists N s.t. $n>N$, $s_n > -M \iff (-s_n) < M$. Thus $\lim_{n\to\infty} -s_n = -\infty$.

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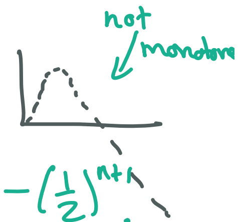
Next time: monotone and Cauchy sequences

Def (increasing/decreasing/monotone sequences):

A sequence s_n is increasing if $s_n \le s_{n+1}\ \forall n$.

A sequence s_n is decreasing if $s_n \ge s_{n+1}\ \forall n$.

A sequence s_n is monotone if it is either increasing or decreasing.



$$\frac{1}{2} \le 1 \iff \left(\frac{1}{2}\right)^{n+1} \le \left(\frac{1}{2}\right)^{n} \iff -\left(\frac{1}{2}\right)^{n} \le -\left(\frac{1}{2}\right)^{n+1} \iff \underbrace{1-\left(\frac{1}{2}\right)^{n}}_{a_n} \le \underbrace{1-\left(\frac{1}{2}\right)^{n+1}}_{a_{n+1}}$$

Ex: $a_n = 1 - \left(\frac{1}{2}\right)^n$ is increasing

$b_n = \sqrt{n}$ is increasing

$c_n = (-1)^n$ is not monotonic

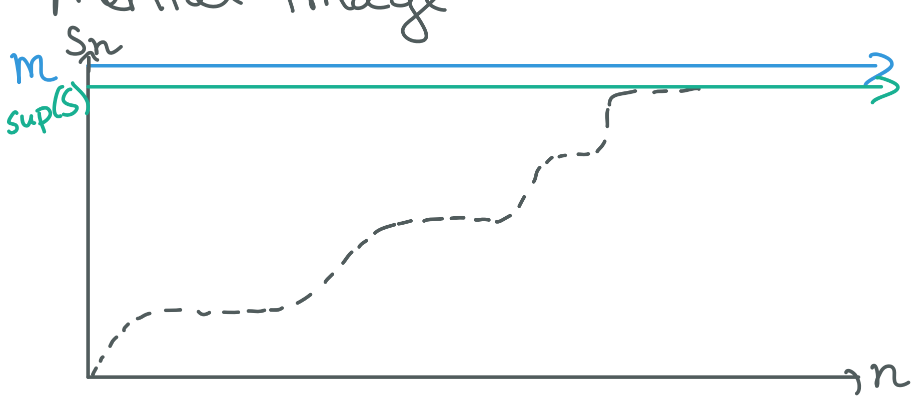
If $m \ge n$, then $m = n+k$ for $k \in \mathbb{Z}$, $k \ge 0$. It suffices to show $s_n \le s_{n+k}$ for all $k \in \mathbb{Z}$, $k \ge 0$. Base case ($k=0$) is true since $s_n = s_{n+0}$. Inductive step: Assume $s_n \le s_{n+k}$. By defn of increasing $s_{n+k} \le s_{n+k+1}$, so $s_n \le s_{n+k+1}$.

Remark: If s_n is increasing, then $s_n \le s_m$ whenever $n \le m$.

MAJOR THEOREM #3

Thm: All bounded, monotone sequences converge.

Mental image:



$S = \{s_n : n \in \mathbb{N}\}$

Pf: Suppose s_n is a bounded, increasing sequence. Define $S = \{s_n : n \in \mathbb{N}\}$.

$\swarrow \exists M$ s.t. $|s_n| \le M \; \forall n \in \mathbb{N}$

Since s_n is a bounded sequence, we have

$\swarrow -M \le t \le M \; \forall t \in S$

that S is a bounded set. By defn of $\mathbb{R}$, S has a supremum. We will show $\lim_{n\to\infty} s_n = \sup(S)$.

Fix $\varepsilon > 0$. Since $\sup(S)$ is an upper bound for S, $\sup(S) \ge s_n \; \forall n \in \mathbb{N}$ and

$$\sup(S) + \varepsilon > s_n \quad \forall n \in \mathbb{N}. \qquad (*)$$

Since $\sup(S)$ is the least upper bound, $\sup(S) - \varepsilon$ is not an upper bound of S, that is there exists $N \in \mathbb{N}$ s.t. $s_N > \sup(S) - \varepsilon$. Since s_n is increasing,

$$s_n > \sup(S) - \varepsilon \quad \forall n > N. \qquad (**)$$

Combining $(*) + (**)$, we have $\forall n > N$, $|s_n - \sup(S)| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\lim_{n\to\infty} s_n = \sup(S)$.

Now suppose s_n is a bounded, decreasing sequence. Then $-s_n$ is a bounded incr. sequence. Thus $\lim_{n\to\infty} -s_n = c \in \mathbb{R}$. By limit theorem, $\lim_{n\to\infty} s_n = (\lim_{n\to\infty} -s_n)(-1) = -c$. Hence s_n converges. $\square$