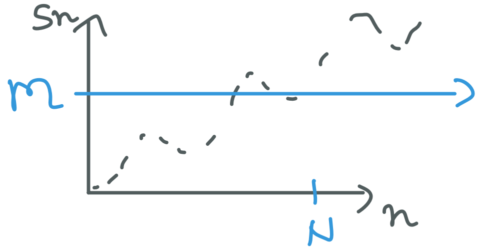
Lecture 7

Def (diverges to $+\infty$ or $-\infty$): A sequence s_n diverges to $+\infty$ if for all $M>0$ there exists $N \in \mathbb{R}$ so that $n>N$ ensures $s_n>M$. We write $\lim_{n\to\infty} s_n = +\infty$.

Likewise, a sequence s_n diverges to $-\infty$ if for all $M<0$ there exists N so that $n>N$ ensures $s_n<M$.
We write $\lim_{n\to\infty} s_n = -\infty$.



Remark:

- If s_n diverges to $\pm\infty$, it does not converge.
- We will say that s_n "has a limit" or "the limit of s_n exists" if either
  1. s_n converges $\lim_{n\to\infty} s_n \in \mathbb{R}$
  2. s_n diverges to $\pm\infty$ $\lim_{n\to\infty} s_n \in \{+\infty, -\infty\}$

Case 1: $\lim_{n\to\infty} t_n = t,\ t>0$
Case 2: $\lim_{n\to\infty} t_n = +\infty$

Thm: Suppose $\lim_{n\to\infty} s_n = +\infty$ and $\lim_{n\to\infty} t_n > 0$.
Then, $\lim_{n\to\infty} s_n t_n = +\infty$.

Thm: Suppose s_n is a sequence of positive real numbers. Then $\lim_{n\to\infty} s_n = +\infty \iff \lim_{n\to\infty} \frac{1}{s_n} = 0$.

Thm: If $\lim_{n\to\infty} s_n = +\infty$, then $\lim_{n\to\infty} (-s_n) = -\infty$.

Def: (increasing/decreasing/monotone)
A sequence s_n is increasing if $s_n \le s_{n+1}$ $\forall n$.
A sequence s_n is decreasing if $s_n \ge s_{n+1}$ $\forall n$.
A sequence s_n is monotone if it is either increasing or decreasing

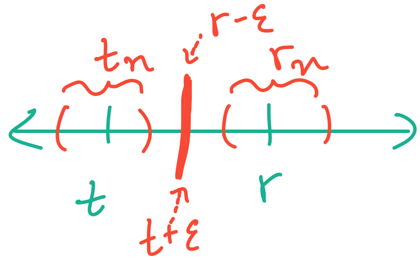
Thm: All bounded monotone sequences converge

Example: HW4, Q11

<u>Lemma</u>: If r_n and t_n are sequences whose limits exist and $r_n \le t_n \ \forall n \in \mathbb{N}$, then

$$\lim_{n\to\infty} r_n \le \lim_{n\to\infty} t_n.$$

Pf:



Case 1: $\lim_{n\to\infty} r_n = -\infty$

The result is immediate.

Case 2: $\lim_{n\to\infty} t_n = +\infty$

The result is immediate.

Case 3: $\lim_{n\to\infty} r_n = r$, $r \in \mathbb{R}$

↳ Case 3a: $\lim_{n\to\infty} t_n = t$, $t \in \mathbb{R}$

Assume, for the sake of contradiction, that $t < r$. Let $\varepsilon = \frac{r-t}{2} > 0$. There exists N_t, N_r so that $n > N_t$ ensures $|t_n - t| < \varepsilon$ and $n > N_r$ ensures $|r_n - r| < \varepsilon$. Let $N = \max\{N_t, N_r\}$. Then $n > N$ ensures:

$$t_n < t + \varepsilon = r - \varepsilon < r_n$$

This contradicts that $r_n \le t_n \ \forall n \in \mathbb{N}$.

↳ Case 3b: Need to rule out $\lim_{n\to\infty} t_n = -\infty$.

Case 4: $\lim_{n\to\infty} r_n = +\infty$. Need to show $\lim_{n\to\infty} t_n = +\infty$.