Lecture 8

In fact, even unbounded monotone sequences always have a limit.

Thm: If s_n is an unbounded, increasing sequence, then $\lim s_n = +\infty$. If s_n is an unbounded, decreasing sequence, then $\lim s_n = -\infty$.

Pf: Suppose s_n is an unbounded, increasing sequence. Fix $M > 0$. Since s_n is unbounded ($\nexists A > 0$ s.t. $|s_n| < A \ \forall n \in \mathbb{N}$) and s_n is increasing ($, s_1 \leq s_n \ \forall n \in \mathbb{N}$), $S = \{s_n : n \in \mathbb{N}\}$ is not bounded above. Thus, $\exists\, N$ s.t. $s_N > M$. Since s_n is increasing, $s_n \geq s_N > M$ $\forall\, n > N$. Since $M > 0$ was arbitrary, this shows $\lim_{n\to\infty} s_n = +\infty$.

Now, suppose s_n is an unbounded, decreasing sequence. Then $-s_n$ is an unbounded, increasing sequence, so $\lim_{n\to\infty} -s_n = +\infty$. Thus, by theorem from last class, $\lim_{n\to\infty} s_n = -\infty$. $\square$

In summary, if s_n is monotone

$$\lim_{n\to\infty} s_n = \begin{cases} s \text{ for } s \in \mathbb{R} & \text{if } s_n \text{ is bdd} \\ +\infty & \text{if } s_n \text{ is unbdd above} \\ -\infty & \text{if } s_n \text{ is unbdd below} \end{cases}$$

s_n is unbdd above if $S = \{s_n : n \in \mathbb{N}\}$ doesn't have an upper bound

Therefore, for any monotone sequence s_n, $\lim s_n$ exists.

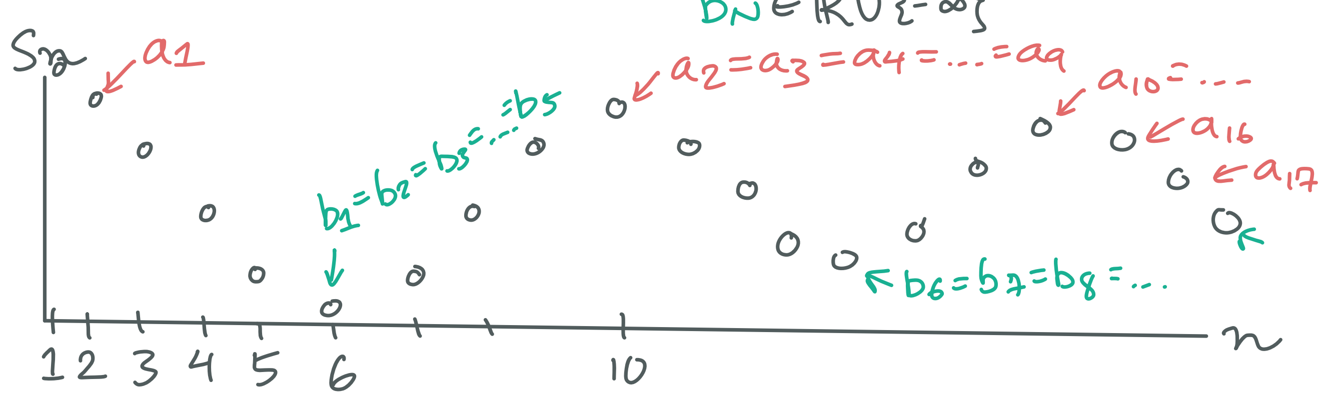
In general, we can think of plenty examples of sequences whose limits don't exist.

It turns out that there is a generalization of the idea of limit that <u>does</u> always exist.

This should remind you of the fact that the maximum of a set S doesn't always exist, but $\sup(S)$ always exist.

Def (limsup / liminf): For any sequence s_n,

- $\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \overbrace{\sup\{s_n : n>N\}}^{a_N \in \mathbb{R}\cup\{+\infty\}}$
- $\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \underbrace{\inf\{s_n : n>N\}}_{b_N \in \mathbb{R}\cup\{-\infty\}}$



FACT: $S \subseteq T$

$\sup(S) \leq \sup(T)$

Why does $\limsup_{n\to\infty} s_n$ and $\liminf_{n\to\infty} s_n$ always exist?

- $\limsup_{n\to\infty} s_n = \lim_{N\to\infty} a_N$ always exists because a_N is a decreasing sequence. Why?

  Because $\{s_n : n > N\} \supseteq \{s_n : n > N+1\}$, so

  $a_N = \sup\{s_n : n > N\} \geq \sup\{s_n : n > N+1\} = a_{N+1}$

  for all $N \in \mathbb{N}$.

- $\liminf_{n\to\infty} s_n = \lim_{N\to\infty} b_N$ always exists because b_N is an increasing sequence. Why?

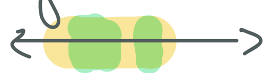
  Because $\{s_n : n > N\} \supseteq \{s_n : n > N+1\}$, so

  $b_N = \inf\{s_n : n > N\} \leq \inf\{s_n : n > N+1\} = b_{N+1}$

  for all $N \in \mathbb{N}$.

FACT: $S \subseteq T$

$\inf(T) \leq \inf(S)$



Examples:

$$\limsup_{n\to\infty} (-1)^n = \lim_{N\to+\infty} \overbrace{\sup\{(-1)^n : n>N\}}^{=\{-1,1\}} = \lim_{N\to\infty} 1 = 1$$

$$\liminf_{n\to\infty} (-1)^n = \lim_{N\to+\infty} \inf\{(-1)^n : n>N\} = \lim_{N\to-\infty} -1 = -1$$

$$\limsup_{n\to\infty} s_n \neq \liminf_{n\to\infty} s_n$$

$$\limsup_{n\to\infty} \frac{1}{n^2} = \lim_{N\to\infty} \sup\left\{\frac{1}{n^2} : n>N\right\} = \lim_{N\to\infty} \frac{1}{(N+1)^2} = 0$$

$$\liminf_{n\to\infty} \frac{1}{n^2} = \lim_{N\to\infty} \inf\left\{\frac{1}{n^2} : n>N\right\} = \lim_{N\to\infty} 0 = 0$$

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n$$

It turns out

$\lim_{n\to\infty} s_n$ exists $\iff \limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n$.

Thm: Given a sequence s_n,

$\lim_{n\to\infty} s_n$ exists $\iff \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.

Furthermore, if either of these equivalent conditions holds, $\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.