Lecture 8 - Highlights

<u>Thm</u>: If s_n is an unbounded, increasing sequence, then $\lim s_n = +\infty$. If s_n is an unbounded, decreasing sequence, then $\lim s_n = -\infty$

In summary, if s_n is monotone

$$\lim_{n\to\infty} s_n = \begin{cases} s \text{ for } s \in \mathbb{R} & \text{if } s_n \text{ is bdd} \\ +\infty & \text{if } s_n \text{ is unbdd above} \\ -\infty & \text{if } s_n \text{ is unbdd below} \end{cases}$$

Therefore, for any monotone sequence s_n, $\lim s_n$ exists.

Def (limsup/liminf): For any sequence s_n,

- $\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \overbrace{\sup\{s_n : n>N\}}^{a_N}$
- $\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \underbrace{\inf\{s_n : n>N\}}_{b_N}$

<u>Thm</u>: Given a sequence s_n,

$$\lim_{n\to\infty} s_n \text{ exists} \iff \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n.$$

Furthermore, if either of these equivalent conditions holds, $\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.

Question ~~5*~~ 3 on Practice Midterm 1

Consider a nonempty set $A \subseteq \mathbb{R}$.

(a) Suppose A is bounded above. Prove that there exists a sequence a_n, satisfying

$$\{a_n : n \in \mathbb{N}\} \subseteq A$$

and

$$\sup A - \frac{1}{n} \le a_n \le \sup A \text{ for all } n \in \mathbb{N}.$$

(Hint: Prove the result by contradiction, using the fact that $\sup A - \frac{1}{n}$ *cannot* be an upper bound.)

(b) Prove that the sequence you found in the previous part satisfies $\lim_{n\to\infty} a_n = \sup A$.

(c) Now suppose A is not bounded above. Prove that there exists a sequence a_n satisfying

$$\{a_n : n \in \mathbb{N}\} \subseteq A$$

and

$$a_n \ge n \text{ for all } n \in \mathbb{N}.$$

(d) Prove that the sequence you found in the previous part satisfies $\lim_{n\to+\infty} a_n = \sup A$

In summary, you have proved the following important result: for any nonempty set $A \subseteq \mathbb{R}$, we may always find a sequence of elements a_n in A so that $\lim_{n\to+\infty} a_n = \sup A$.

a) Proof: Suppose A is a nonempty set $A \subseteq \mathbb{R}$.

Suppose A is bounded above, then $\exists m \in \mathbb{R}$ s.t. $a \le m$ ~~$\forall n \in \mathbb{N}$~~ $\forall a \in A$.

~~By contradiction, there does not exist a sequence a_n satisfying $\{a_n : n \in \mathbb{N}\} \subseteq A$ and $\sup A - \frac{1}{n} \le a_n \le \sup A$ $\forall n \in \mathbb{N}$. This is equivalent of showing $\forall \{a_n : n \in \mathbb{N}\} \subseteq A$, $\exists n \in \mathbb{N}$ st. $a_n < \sup A - \frac{1}{n}$ or $a_n > \sup A$.~~

Since A is nonempty and bounded above, $\sup(A)$ exists.

Since by definition, $\sup(A)$ is the least upper bound, and $\forall n \in \mathbb{N}$, $\frac{1}{n} > 0$, $\sup A - \frac{1}{n}$ cannot be an upper bound, of A ~~meaning $\exists n \in \mathbb{N}$ s.t. $a_n > \sup A - \frac{1}{n}$~~ Thus $\forall n \in \mathbb{N}$, $\exists a \in A$ so that $\sup A - \frac{1}{n} < a$. Define a sequence so that its nth element is such an a.

~~Thus $a_n < \sup A - \frac{1}{n}$ is not true ①~~

Therefore, we have a sequence a_n with $\{a_n : n \in \mathbb{N}\} \subseteq A$ and $\sup A - \frac{1}{n} < a_n$.

Also, by the definition of supremum, $a_n \le \sup(A)$ $\forall n \in \mathbb{N}$. ②

~~But ①, ② contradict with the assumption $a_n < \sup A - \frac{1}{n}$ or $a_n > \sup A$.~~

Thus, $\exists$ a sequence a_n satisfying $\{a_n : n \in \mathbb{N}\} \subseteq A$ and ~~$a_n \ge n$ $\forall n \in \mathbb{N}$~~ $\sup A - \frac{1}{n} < a_n \le \sup A$. ∎

b) Proof: Let $s_n = \sup(A) - \frac{1}{n}$, $t_n = \sup(A)$. Then $s_n \le a_n \le t_n$.

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\sup(A) - \frac{1}{n}\right) = \lim_{n\to\infty}(\sup(A)) + \lim_{n\to\infty}\left(-\frac{1}{n}\right)$$

[limit of sum is sum of limit]

$$= \lim_{n\to\infty}(\sup(A)) + 0 \text{ [Basic example of limits] } = \sup(A).$$

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty}\sup(A) = \sup(A).$$

Since $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = \sup(A)$. By squeeze theorem (Q3), $s_n \le a_n \le t_n$

$$\Rightarrow \lim_{n\to\infty} a_n = \sup(A). \blacksquare$$

c) Proof: Suppose A is not bounded above, we WTS $\exists a_n$ s.t. $\{a_n : n \in \mathbb{N}\} \subseteq A$ and $a_n \geq n \ \forall n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. let $a_n = n^2$. then a_n is not bounded above.

Since $n \in \mathbb{N} \subseteq \mathbb{R}$, $n^2 \in \mathbb{R}$, thus $\{a_n : n \in \mathbb{N}\} \subseteq A$

Since $n \in \mathbb{N}$, $n \geq 1$, $\Rightarrow n \cdot n \geq 1 \cdot n \Rightarrow n^2 \geq n \Leftrightarrow a_n \geq n \ \forall n \in \mathbb{N}$.

Therefore, such a_n exists. $\blacksquare$

d) Proof: Since A is not bounded above, by definition, $\sup(A) = +\infty$.

Fix $m > 0$. let $m = N$, $\forall n > N$, $a_n \geq n > N = m$. Thus $a_n > m$

Since $n > N$ ensures $a_n > m$, a_n diverges to $+\infty$, $\lim_{n \to \infty} a_n = +\infty$.

Thus, $\lim_{n \to \infty} a_n = +\infty = \sup(A)$. $\blacksquare$