Lecture 9

Thm: Given a sequence s_n,

$$\lim_{n\to\infty} s_n \text{ exists} \iff \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n.$$

Furthermore, if either of these equivalent conditions holds, $\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.

Remarks:

① If $s < M$ $\forall s \in S$, then $\sup(S) \leq M$.

② $b_N = \inf\{s_n : n > N\} \leq \sup\{s_n : n > N\} = a_N$

③ If r_n and t_n are sequences whose limits exist and $r_n \leq t_n$ $\forall n \in \mathbb{N}$, then $\lim_{n\to\infty} r_n \leq \lim_{n\to\infty} t_n$.

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} b_N \leq \lim_{N\to\infty} a_N = \limsup_{n\to\infty} s_n$$

④ $$\limsup_{n\to\infty}(-s_n) = \lim_{N\to\infty} \sup\{-s_n : n > N\}$$
$$= \lim_{N\to\infty} -\inf\{s_n : n > N\}$$
$$= -\lim_{N\to\infty} \inf\{s_n : n > N\}$$
$$= -\liminf_{n\to\infty} s_n$$

Similarly, $\liminf_{n\to\infty} -s_n = -\limsup_{n\to\infty} s_n$.

Pf:

$\boxed{\text{Suppose } \lim s_n \text{ exists. WTS } \liminf s_n = \limsup s_n = \lim s_n.}$

$\boxed{\text{CASE 1}}$ $\boxed{\text{Suppose } \lim s_n = -\infty}$, so for all $M<0$, there exists N_0 s.t. $n > N_0$, $s_n < M$. Thus $a_{N_0} = \sup\{s_n : n > N_0\} \leq M$. Since a_N is a decreasing sequence, $a_N \leq a_{N_0} \leq M \ \forall \ N \geq N_0$. Since $M<0$ was arbitrary, by the definition of divergence to $-\infty$, we have $\lim_{N\to\infty} a_N = -\infty$. Thus $\liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n = \lim_{N\to\infty} a_N = -\infty$, so $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = \lim_{n\to\infty} s_n = -\infty$.

$\boxed{\text{CASE 2}}$ $\boxed{\text{Suppose } \lim s_n = +\infty}$ Then $\lim_{n\to\infty} -s_n = -\infty$. By previous case, $\liminf_{n\to\infty} -s_n = \limsup_{n\to\infty} -s_n = \lim_{n\to\infty} -s_n = -\infty$. Thus, $-\limsup_{n\to\infty} s_n = -\liminf_{n\to\infty} s_n = -\lim_{n\to\infty} s_n = -\infty$. Multiplying by -1, $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \lim_{n\to\infty} s_n = +\infty$.

$\boxed{\text{CASE 3}}$ $\boxed{\text{Suppose } \lim s_n = s, \text{ for } s \in \mathbb{R}}$ Fix $\varepsilon > 0$. By defn of convergence, $\exists \ N_0$ s.t. $n > N_0$ implies $|s_n - s| < \varepsilon \iff s - \varepsilon < s_n < s + \varepsilon$. Hence $a_{N_0} = \sup\{s_n : n > N_0\} \leq s+\varepsilon$, and since a_N is decreasing, $N > N_0$ ensures $a_N \leq a_{N_0} \leq s + \varepsilon$. Likewise, $b_{N_0} = \inf\{s_n : n > N_0\} \geq s - \varepsilon$, and since b_N is increasing, $N > N_0$ ensures $b_N \geq b_{N_0} \geq s - \varepsilon$.

Thus, $N > N_0$ ensures $s-\varepsilon \leq b_N \leq a_N \leq s+\varepsilon$.
Since $\varepsilon > 0$ was arbitrary, $\lim_{N\to\infty} b_N = \lim_{N\to\infty} a_N = s$,
that is $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$.

CLAIM: Consider a sequence s_n and $s \in \mathbb{R}$. If $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - s| \leq \varepsilon$, then s_n converges to s.

Pf: Fix $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$. By hypothesis, $\exists\, N \in \mathbb{R}$ s.t. $n > N$ ensures $|s_n - s| \leq \frac{\varepsilon}{2} < \varepsilon$.
Thus s_n converges to s.

$\boxed{\text{Now suppose } \liminf s_n = \limsup s_n \quad \text{WTS } \lim s_n \text{ exists and } \lim s_n = \liminf s_n = \limsup s_n.}$

$\boxed{\text{CASE 1}}$ $\boxed{\text{Suppose } \liminf s_n = \limsup s_n = -\infty}$ By defn of limsup, $\lim_{N\to\infty} a_N = -\infty$. Fix $M<0$. There exists N_0 s.t. $N > N_0$, $\sup\{s_n : n > N\} = a_N < M$. Let $N_1 = \lceil N_0 \rceil + 1 = \min\{m \in \mathbb{N} : m \geq N_0\} + 1$, so $\sup\{s_n : n > N_1\} = a_{N_1} < M$. Thus $s_n < M$ for all $n > N_1$. Since $M<0$ was arbitrary, $\lim_{n\to\infty} s_n = -\infty$.

$\boxed{\text{CASE 2}}$ $\boxed{\liminf s_n = \limsup s_n = +\infty}$. Then $\liminf_{n\to\infty} -s_n = \limsup_{n\to\infty} -s_n = -\infty$. By what we just showed, $\lim_{n\to\infty} -s_n = -\infty$, so $\lim_{n\to\infty} s_n = +\infty$.

$\boxed{\text{CASE 3}}$ $\boxed{\liminf s_n = \limsup s_n = s \text{ for } s \in \mathbb{R}}$ Thus $\lim_{N\to\infty} a_N = \lim_{N\to\infty} b_N = s$. Fix $\varepsilon > 0$. There exists N_a and N_b so that $N > N_a$ ensures $|a_N - s| < \varepsilon$ and $N > N_b$ ensures $|b_N - s| < \varepsilon$. Define $N_0 = \lceil \max\{N_a, N_b\} \rceil + 1$. Then

$b_{N_0} > s - \varepsilon$ and $a_{N_0} < s + \varepsilon$. Therefore for all $n > N_0$,

$$s - \varepsilon < b_{N_0} = \inf\{s_n : n > N_0\} \leq s_n \leq \sup\{s_n : n > N_0\} = a_{N_0} < s + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\lim_{n \to \infty} s_n = s$. $\square$