Midterm 1 Solutions

S is bounded above if $\exists M \in \mathbb{R}$ so that $s < M$ for all $s \in S$

1

a

b

By defn of $A+B$ and the supremum, $\sup(A+B)$ is an upper bound for $A+B$, so $a+b \leq \sup(A+B) \iff a \leq \sup(A+B) - b$ for all $a \in A$, $b \in B$. Thus, for all $b \in B$, $\sup(A+B) - b$ is an upper bound for A.

c

By defn of $\sup(A)$ as the least upper bound for A, $\sup(A) \leq \sup(A+B) - b$ for all $b \in B$. Thus

$$b \leq \sup(A+B) - \sup(A)$$

for all $b \in B$

This shows $\sup(A+B) - \sup(A)$ is an upper bound for B, hence $\sup(B) \leq \sup(A+B) - \sup A$.

This shows $\sup(A) + \sup(B) \leq \sup(A+B)$.

To see the other inequality, note that, since $\sup(A) \geq a \quad \forall a \in A$ and $\sup(B) \geq b \quad \forall b \in B$,

$$\sup(A) + \sup(B) \geq a+b \quad \forall a \in A, b \in B.$$

Thus $\sup(A) + \sup(B)$ is an upper bound for $A+B$, hence

$$\sup(A) + \sup(B) \geq \sup(A+B).$$

(2)

(a)

A sequence s_n converges to a limit $s \in \mathbb{R}$ if, $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ s.t. $n > N$ ensures $|s_n - s| < \varepsilon$.

2b)
A sequence s_n diverges to $+\infty$ if, for all $M > 0$, there exists N so that $n > N$ ensures $s_n > M$.

2 c) $\lim_{n\to\infty} t_n$ exists

WTS $\lim_{n\to\infty} |t_n| = |\lim_{n\to\infty} t_n|$.

$\boxed{\text{Case 1}}$: $\lim_{n\to\infty} t_n = +\infty$. WTS $\lim_{n\to\infty} |t_n| = +\infty$.

Fix $M > 0$. $\exists N$ s.t. $n > N$ ensures $t_n > M$. Since $|t_n| \geq t_n$, $n > N$ ensures $|t_n| > M$. Thus $\lim_{n\to\infty} |t_n| = +\infty$.

$\boxed{\text{Case 2}}$: $\lim_{n\to\infty} t_n = t$ for $t \in \mathbb{R}$. WTS $\lim_{n\to\infty} |t_n| = |t|$.

Fix $\varepsilon > 0$. $\exists N$ s.t. $n > N$ ensures $|t_n - t| < \varepsilon$.

By reverse triangle inequality $||t_n| - |t|| \leq |t_n - t|$.

Thus, $n > N$ ensures $||t_n| - |t|| < \varepsilon$. Thus, $\lim_{n\to\infty} |t_n| = |t|$.

$\boxed{\text{Case 3}}$: $\lim_{n\to\infty} t_n = -\infty$. WTS $\lim_{n\to\infty} |t_n| = +\infty$.

By result from class, $\lim_{n\to\infty} -t_n = +\infty$. By part ⓐ, $\lim_{n\to\infty} |-t_n| = +\infty$. Since $|-t_n| = |t_n|$, this gives the result.

2d) $t_n = (-1)^n$

3) See practice midterm Q3

4)
(1) Note that $S = (1/2,1]$
(i) yes, $\sup(S) = 1$
(ii) no, $\inf(S) = 1/2$

(2)
(i) (b)
(ii) (c)