Midterm 2 Review

Def (maximum, minimum): Suppose $S \subseteq F$, where F is an ordered field.

- If there exists $s_0 \in S$ satisfying $s_0 \geq s$ for all $s \in S$, then s_0 is the <u>maximum</u> of S and write $s_0 = \max(S)$.
  "s_0 is the largest element in the set"
- If there exists $s_0 \in S$ satisfying $s_0 \leq s$ for all $s \in S$, then s_0 is the <u>minimum</u> of S and write $s_0 = \min(S)$.
  "s_0 is the smallest element in the set"

Def: (bounded above/below): Suppose $S \subseteq F$ for an ordered field F.

- If there exists $M \in F$ satisfying $s \leq M$ $\forall s \in S$, then S is <u>bounded above</u> and M is an <u>upper bound</u> of S.
- If there exists $m \in F$ satisfying $s \geq m$ $\forall s \in S$, then S is <u>bounded below</u> and m is a <u>lower bound</u> of S.
- If S is bounded above and bounded below, then S is <u>bounded</u>.

<u>Def</u> (supremum / infimum): Consider an ordered field F.

- If $S \subseteq F$ is bounded above and there exists $M_0 \in F$ satisfying...
  - (a) M_0 is an upper bound of S
  - (b) if M is an upper bound of S, then $M_0 \le M$

  we say M_0 is the <u>supremum</u> of S and write $M_0 = \sup(S)$.

  "M_0 is the least upper bound"

- If $S \subseteq F$ is bounded below and there exists $m_0 \in F$ satisfying...
  - (a) m_0 is a lower bound of S
  - (b) if m is a lower bound of S, then $m_0 \ge m$

  we say m_0 is the <u>infimum</u> of S and write $m_0 = \inf(S)$.

  "m_0 is the greatest lower bound"

<u>Thm</u>: Consider an ordered field F and $S \subseteq F$.

(i) If $\max(S)$ exists, then $\sup(S) = \max(S)$.

(ii) If $\min(S)$ exists, then $\inf(S) = \min(S)$.

Def: (real numbers): The set of real numbers is the ordered field containing $\mathbb{Q}$ with the property that every nonempty subset $S \subseteq \mathbb{R}$ that is bounded above has a supremum.

MAJOR RESULT #1

Thm (Archimedean Property): If $a, b \in \mathbb{R}$ satisfy $a > 0$ and $b > 0$, then there exists $n \in \mathbb{N}$ so that $na > b$.

spoon bathtub

MAJOR THEOREM #2

Thm ($\mathbb{Q}$ is dense in $\mathbb{R}$): If $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ satisfying $a < r < b$.

Def (Unbounded above/below): Suppose $S \subseteq \mathbb{R}$ is nonempty.

- If S is not bounded above, write $\sup(S) = +\infty$.
- If S is not bounded below, write $\inf(S) = -\infty$.

Def (convergence):

- A sequence s_n of real numbers **converges** to some $s \in \mathbb{R}$ provided that

  [for all $\varepsilon > 0$, there exists $N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - s| < \varepsilon$.]

- The number s is the **limit** of s_n, and we write $\lim_{n\to\infty} s_n = s$ or $s_n \to s$.

- A sequence that does not converge to any $s \in \mathbb{R}$ it is said to **diverge**.

Thm (limit of sum is sum of limits): If s_n and t_n are convergent sequences, $\lim_{n\to\infty}(s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$.

Thm (limit of product is product of limits): If s_n and t_n are convergent sequences, $\lim_{n\to\infty} s_n t_n = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right)$

Thm (limit of quotient is quotient of limits): If s_n and t_n are convergent sequences, $s_n \neq 0$ for all n, and $\lim_{n\to\infty} s_n \neq 0$, then

$$\lim_{n\to\infty}\left(\frac{t_n}{s_n}\right) = \frac{\lim_{n\to\infty} t_n}{\lim_{n\to\infty} s_n}.$$

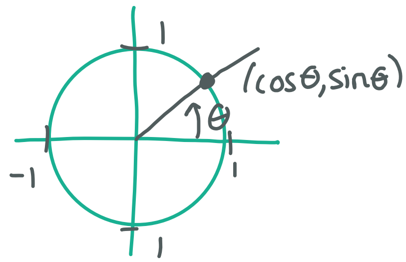
Thm (basic examples):

(a) $\lim_{n\to\infty} \left(\frac{1}{n}\right)^p = 0$ if $p>0$

(b) $\lim_{n\to\infty} a^n = 0$ if $|a|<1$

(c) $\lim_{n\to\infty} n^{1/n} = 1$

(d) $\lim_{n\to\infty} a^{1/n} = 1$ if $a>0$



Def (diverges to $+\infty$ or $-\infty$): A sequence s_n diverges to $+\infty$ if for all $M>0$ there exists $N\in\mathbb{R}$ so that $n>N$ ensures $s_n>M$. We write $\lim_{n\to\infty} s_n = +\infty$.

Likewise, a sequence s_n diverges to $-\infty$ if for all $M<0$ there exists N so that $n>N$ ensures $s_n<M$. We write $\lim_{n\to\infty} s_n = -\infty$.

Def: (increasing/decreasing/monotone)
A sequence s_n is increasing if $s_n \le s_{n+1}$ $\forall n$.
A sequence s_n is decreasing if $s_n \ge s_{n+1}$ $\forall n$.
A sequence s_n is monotone if it is either increasing or decreasing

Thm: All bounded monotone sequences converge

Thm: If s_n is an unbounded, increasing sequence, then $\lim s_n = +\infty$. If s_n is an unbounded, decreasing sequence, then $\lim s_n = -\infty$

Def (limsup/liminf): For any sequence s_n,

- $\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \overbrace{\sup\{s_n : n > N\}}^{a_N \in \mathbb{R}\cup\{+\infty\}}$
- $\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \underbrace{\inf\{s_n : n > N\}}_{b_N \in \mathbb{R}\cup\{-\infty\}}$

Thm: Given a sequence s_n,

$$\lim_{n\to\infty} s_n \text{ exists} \iff \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n.$$

Furthermore, if either of these equivalent conditions holds, $\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.

Def (Cauchy sequence): A sequence s_n is a Cauchy sequence if for all $\varepsilon > 0$, there exists $N \in \mathbb{R}$ s.t. $m,n > N$ ensures $|s_n - s_m| < \varepsilon$

MAJOR THEOREM #4

Thm: A sequence is convergent iff it is Cauchy.

Def: (subsequential limit) A subsequential limit of a sequence s_n is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of s_n.

Thm: If a sequence s_n converges to a limit s, then every subsequence also converges to s.

(*)

Thm (main subsequences theorem)
Let s_n be a sequence of real numbers.

(a) Let $t \in \mathbb{R}$

[The set $\{n : |s_n - t| < \varepsilon\}$ is infinite for all $\varepsilon > 0$]

if and only if

[t is a subsequential limit of s_n.]

(b) s_n is unbounded above $\iff$ $+\infty$ is a subseq. limit.

(c) s_n is unbounded below $\iff$ $-\infty$ is a subseq. limit.

Thm: Every sequence s_n has a monotonic subsequence.

↙ MAJOR THEOREM 5

Thm (Bolzano-Weierstrass): Every bounded sequence has a convergent subsequence.

Thm: Let S denote the set of subsequential limits of s_n. Then $\limsup s_n = \max(S)$ and $\liminf s_n = \min(S)$

Leslie's Thm: Suppose $\lim_{n\to\infty} s_n$ exists. For any subsequence s_{n_k}, $\lim_{k\to\infty} s_{n_k} = \lim_{n\to\infty} s_n$.

Pf: If $\lim_{n\to\infty} s_n = s$ for $s \in \mathbb{R}$. This is a consequence of (*). On the other hand suppose $\lim_{n\to\infty} s_n = +\infty$. Fix a subsequence s_{n_k}. We will show $\lim_{k\to\infty} s_{n_k} = +\infty$. Fix $M>0$ arbitrary. There exists N s.t. $n>N$ ensures $s_n > M$. Recall that $n_k \geq k$. Thus, if $k>N$, $n_k > N$ and $s_{n_k} > M$. This shows $\lim_{k\to\infty} s_{n_k} = +\infty$. (Similar for divergence to $-\infty$.)