Math 117: Midterm 2A

Tuesday, March 5, 2024

Name: ____

Student ID #: ____

Signature: ____

This is a closed-book and closed-note examination. Calculators are not allowed. Please show your work in the space provided. I will provide scratch paper—other forms of scratch paper are not permitted. If you continue a problem on the back of a page, please write "continued on back". Partial credit will be given for partial answers. You have 1 hour and 15 minutes.

Question	Points	Score
1	24 28	
2	32 28	
Total	56	

Question 1 (28 points)

Suppose s_n and t_n are bounded sequences.

(a) State the theorem that the limit of the sum is the sum of the limits.

(b) Prove that, for all $N \in \mathbb{N}$,

$$\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}.$$

(c) Prove that $\limsup s_n + t_n \leq \limsup s_n + \limsup t_n$.

(d) Give an examples of bounded sequences s_n and t_n for which $\limsup s_n + t_n < \limsup s_n + \limsup t_n$.

(a) If s_n and t_n are convergent sequences, then $\lim_{n\to\infty}(s_n+t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$

(b) For any $n>N$, $s_n \leq \sup\{s_n : n>N\}$ and $t_n \leq \sup\{t_n : n>N\}$, so

$s_n + t_n \leq \sup\{s_n : n>N\} + \sup\{t_n : n>N\}$.

Note that the right hand side is a real number, sinse s_n and t_n are bounded sequences. Thus, the right hand side is an upper bound for $\{s_n+t_n : n>N\}$. The fact that the suprenum is the least upper bound gives the result.

(c) Let $a_N = \sup\{s_n+t_n : n>N\}$, $b_N = \sup\{s_n : n>N\}$, and $c_N = \sup\{t_n : n>N\}$. By part (b),

$$a_N \leq b_N + c_N \quad \forall N \in \mathbb{N}$$

Furthermore, all three sequences are $\longrightarrow$

bounded and monotone, hence convergent. Thus

$$\limsup_{n\to\infty}(s_n+t_n) = \lim_{N\to\infty} a_N$$

$$\leq \lim_{N\to\infty} (b_N + c_N)$$

$$\overset{(a)}{=} \lim_{N\to\infty} b_N + \lim_{N\to\infty} c_N$$

$$= \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n.$$

(d) $s_n = (-1)^n$, $t_n = (-1)^{n+1}$

Question 2 (28 points)

Consider a sequence s_n.

(a) State the definition of $\limsup s_n$.

(b) If $\limsup |s_n| < +\infty$, prove that s_n is a bounded sequence.

(c) If s_n is a bounded sequence, prove that $\limsup |s_n| < +\infty$.

(d) If $\limsup |s_n| < +\infty$, prove that s_n has a convergence subsequence.

(a) $\limsup\limits_{n\to\infty} s_n = \lim\limits_{N\to\infty} \sup\{s_n : n > N\}$

(b) If $\limsup\limits_{n\to\infty} |s_n| < +\infty$, then $\lim\limits_{N\to\infty} \sup\{|s_n| : n > N\} < +\infty$.

In particular, the sequence $a_N = \sup\{|s_n| : n > N\}$ is convergent, so $a_1 = \sup\{|s_n| : n > 1\} < +\infty$.

Thus $|s_n| \leq \max\{a_1, |s_1|\}$ $\forall n \in \mathbb{N}$, so s_n is bounded.

(c) If s_n is bounded, $\exists\, M > 0$ s.t. $|s_n| < M$ $\forall n \in \mathbb{N}$. Thus $a_N = \sup\{|s_n| : n > N\} \leq M$ for all $N \in \mathbb{N}$.

Hence $\limsup\limits_{n\to\infty} s_n = \lim\limits_{N\to\infty} a_N \leq M < +\infty$

(d) By part (b) s_n is bounded. By Bolzano Weierstrass, all bounded sequences have a convergent subsequence.

Math 117: Midterm 2b

Thursday, March 7th, 2024

Name: ____

Student ID #: ____________________________________

Signature: ______________________________________

This is a closed-book and closed-note examination. Calculators are not allowed. Please show your work in the space provided. I will provide scratch paper—other forms of scratch paper are not permitted. If you continue a problem on the back of a page, please write "continued on back". Partial credit will be given for partial answers. You have 1 hour and 15 minutes.

Question	Points	Score
3	~~28~~ 30	
4	~~16~~ 14	
Total	44	

Question 3 (30 points)

Define a sequence s_n as follows: $s_1 = 1$ and, for $n > 1$, $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2$.

(a) State the theorem that the limit of a quotient is the quotient of the limits.

(b) Prove that $0 \leq s_n \leq 1$ for all $n \in \mathbb{N}$.

(c) Prove that s_n is a decreasing sequence.

(d) Explain why s_n converges.

(e) Use the definition of s_n to find the value of s, where $s = \lim_{n\to+\infty} s_n$.

In part (e), you may assume and use all the main limit theorems from class ("limit of sum is sum of limits", "limit of product is product of limits", "limit of quotient is quotient of limits"), without further justification. You may also use the following fact from the homework without further justification:

Fact: If c_k is a convergent sequence, then $\lim_{k\to+\infty} c_{k+1} = \lim_{k\to+\infty} c_k$.

(a) If s_n and t_n are convergent sequences, $t_n \neq 0$ $\forall n \in \mathbb{N}$, and $\lim_{n\to\infty} t_n \neq 0$, then

$$\lim_{n\to\infty}\left(\frac{s_n}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n}$$

(b) We proceed by induction. It is clear that $0 \leq s_1 \leq 1$. Suppose $0 \leq s_n \leq 1$. Then since $\frac{n}{n+1} \geq 0$, $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2 \geq 0$. Likewise, since $\frac{n}{n+1} \leq 1$ and $s_n^2 \leq s_n \leq 1$, $\left(\frac{n}{n+1}\right) s_n^2 \leq 1$. This gives the result.

(c) Since $0 \leq s_n \leq 1$, $s_n^2 \leq s_n$. Since $0 \leq \frac{n}{n+1} \leq 1$, $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2 \leq 1 \cdot s_n^2 \leq s_n$. This shows it is decreasing.

(d) All bounded monotone sequences converge.

(e) Let $s \in \mathbb{R}$ be the limit of s_n. By Fact,

$$s = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right) s_n^2 .$$

Since s_n is convergent, $\lim_{n\to\infty} s_n^2 = (\lim_{n\to\infty} s_n)(\lim_{n\to\infty} s_n) = s^2$.
Since $\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$ converges to 1, $\lim_{n\to\infty} \left(\frac{n}{n+1}\right) s_n^2 = s^2$.
Thus $s = s^2$. We must have either $s=0$ or $s=1$. Since s_n is decreasing and $s_2 = \frac{2}{3}$, we have $s=0$.

Question 4 (14 points)

Lightning Round!
You do not need to show your work or justify your answers.

(1) Consider the sequence $a_n = \cos\left(\cancel{\frac{n\pi}{2}}\right)$. $\frac{n\pi}{4}$

(i) What is $\limsup_{n\to+\infty} a_n$ and $\liminf_{n\to+\infty} a_n$?

$= -1$ $\qquad$ $= 1$

(ii) Does the sequence have a limit? Is it a Cauchy sequence?

no; no

(2) State whether the following statements are true or false. If they are false, provide a counterexample. You do not need to justify your counterexample.

(i) Every convergent sequence is bounded.

True

(ii) Every monotone sequence is Cauchy.

False, $s_n = n$