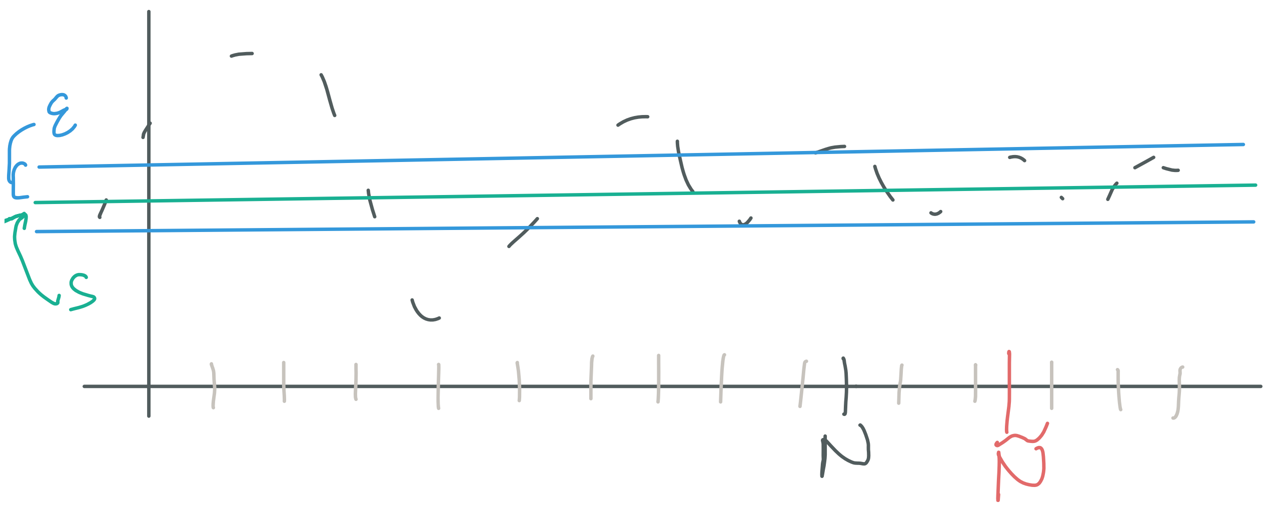
Def (our course): A sequence s_n converges to a limit $s \in \mathbb{R}$ if, $\forall \varepsilon > 0$, $\exists$ $N \in \mathbb{R}$ s.t. $n > N$ ensures $|s_n - s| < \varepsilon$.

Def (alternative): A sequence s_n converges to a limit $s \in \mathbb{R}$ if, $\forall \varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ s.t. $n \geq N$ ensures $|s_n - s| < \varepsilon$.

Lemma: Both definitions are equivalent.

Pf: Suppose $s_n \to s$ in the sense of the first definition. Fix $\varepsilon > 0$. Then $\exists$ $N \in \mathbb{R}$ so $n > N$ ensures $|s_n - s| < \varepsilon$. Let $\tilde{N} = \max\{\lceil N \rceil + 1, 1\}$. Then $\tilde{N} \in \mathbb{N}$, $\tilde{N} > N$. Then $n \geq \tilde{N}$, we must have $n > N$, so $|s_n - s| < \varepsilon$. This shows s_n converges in the sense of the second definition.

Now, suppose $s_n \to s$ in the sense of the alternate definition. Fix $\varepsilon > 0$. Then $\exists$ $N \in \mathbb{N}$ s.t. $n \geq N$ ensures $|s_n - s| < \varepsilon$. Let $\tilde{N} = N$. Then $\tilde{N} \in \mathbb{R}$ and $n > \tilde{N}$ implies $n \geq N$, so $|s_n - s| < \varepsilon$.



Practice Midterm, Q2(c)

WTS $\lim\limits_{n\to\infty} \frac{n-3}{n^2+9} = 0$.

Scratchwork:

$|ab| = |a||b|$

$\left|\frac{c}{d}\right| = \frac{|c|}{|d|}$

$$\left|\frac{n-3}{n^2+9} - 0\right| < \varepsilon \iff \left|\frac{n-3}{n^2+9}\right| < \varepsilon \iff \frac{|n-3|}{n^2+9} < \varepsilon$$

$$\Leftarrow \frac{|n-3|}{n^2} < \varepsilon \Leftarrow \frac{n+3}{n^2} < \varepsilon \Leftarrow \frac{n+3n}{n^2} < \varepsilon \iff \frac{4\cancel{n}}{n^{\cancel{2}}} < \varepsilon$$

$|n-3| \leq n+3$

$|x-y| \leq |x| + |y|$

$$\Updownarrow$$

$$\frac{4}{\varepsilon} < n$$

Warning:

$$\frac{n+3}{n^2} < \varepsilon \Leftarrow \frac{n^2+3n^2}{n^2} < \varepsilon \iff 4 < \varepsilon$$

Ⓒ Fix $\varepsilon > 0$. Let $N = \frac{4}{\varepsilon}$. Then $n > N$ ensures

$$\frac{4}{n} < \varepsilon \iff \frac{4n}{n^2} < \varepsilon \iff \frac{n+3n}{n^2} < \varepsilon \Rightarrow \frac{n+3}{n^2} < \varepsilon \Rightarrow$$

$$\frac{|n-3|}{n^2} < \varepsilon \Rightarrow \left|\frac{n-3}{n^2}\right| < \varepsilon \Rightarrow \left|\frac{n-3}{n^2+9}\right| < \varepsilon \iff \left|\frac{n-3}{n^2+9} - 0\right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this gives the result.

Def (our course): A sequence s_n converges to a limit $s \in \mathbb{R}$ if, $\forall \varepsilon > 0$, $\exists$ $N \in \mathbb{R}$ s.t. $n > N$ ensures $|s_n - s| < \varepsilon$.

(2)(b)

A sequence does not converge to a limit s if, $\exists\, \varepsilon > 0$ s.t. $\forall N \in \mathbb{R}$, $\exists\, n > N$ for which $|s_n - s| \geq \varepsilon$.

② Assume, for the sake of contradiction, that s_n converges to some $s \in \mathbb{R}$. Then, for $\varepsilon = 1$, there exists $N \in \mathbb{R}$ so that $n > N$ ensures

$$
\begin{aligned}
|s_n - s| < 1 &\iff s-1 < s_n < s+1 \\
&\iff s-1 < (n+1)^2 - 2 < s+1 \\
&\iff s+1 < (n+1)^2 < s+3
\end{aligned}
$$

Scratchwork:
$s+3 \le (n+1)^2 \Longleftarrow s+3 \le n$

By basic properties of $\mathbb{R}$, there exists $k \in \mathbb{N}$ s.t. $k > N$ and $k \ge s+3$.

~~By the lemma following the Archimedean Property, $\exists\ m \in \mathbb{N}$ so that $m > s+3$. Let $k = \max(m, N+1)$. Then $k \ge m > s+3$ and $k > N$.~~ The latter ensures: since $k>N$,

$$(k+1)^2 < s+3 \Rightarrow k < k^2+2k+1 < s+3$$

This contradicts $\circledast$. Thus s_n must not converge to any $s \in \mathbb{R}$.