Practice Midterm 1 Solutions

①

ⓐ S is bounded below if there exists $m_0 \in \mathbb{R}$ s.t. $m_0 \leq s$ for all $s \in S$.

ⓑ Suppose $a > 0$ is a lower bound for S. Then $a \leq s$ $\forall s \in S$, which is equivalent to $\frac{1}{s} \leq \frac{1}{a}$ $\forall s \in S$ Furthermore, this is equivalent to $t \leq \frac{1}{a}$ $\forall t \in S'$. Finally, we recognize this is equivalent to $\frac{1}{a}$ being an upper bound for S'.

ⓒ If $\inf S > 0$, then by part ⓑ, we use the fact that $\inf S$ is a lower bound for S to conclude that $\frac{1}{\inf S}$ is an upper bound for S'.

Suppose M is an upper bound for S'. Since $S' \subseteq (0, +\infty)$ is nonempty, M must be strictly positive. By part ⓑ, $\frac{1}{M}$ is a lower bound for S.

Thus, by definition of the greatest lower bound, $\frac{1}{m} \le \inf S$. Thus, $\frac{1}{\inf S} \le m$. This shows that $\frac{1}{\inf S}$ is the least upper bound of S'. Thus $\frac{1}{\inf S} = \sup S'$.

(d) By definition of what $\sup S' = +\infty$ means, it suffices to show S' is not bounded above.

Suppose, for the sake of contradiction that M were an upper bound for S'. Again, since $S' \subseteq (0, +\infty)$ is nonempty, we have $M > 0$. By part (b), $\frac{1}{M} > 0$. is a lower bound for S. This contradicts that $\inf S = 0$ is the greatest lower bound.

Thus, S' is not bounded above.

(2)

(a) A sequence s_n converges to a limit $s \in \mathbb{R}$ if, $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ s.t. $n > N$ ensures $|s_n - s| < \varepsilon$.

(b) A sequence s_n does not converge to a limit $s \in \mathbb{R}$ if $\exists\, \varepsilon > 0$ s.t. for all $N \in \mathbb{R}$, $\exists\, n > N$ s.t. $|s_n - s| \geq \varepsilon$.

(c) Fix $\varepsilon > 0$. Let $N = \frac{4}{\varepsilon}$. Then $n > N$ ensures

$$\frac{4}{n} < \varepsilon \iff \frac{4n}{n^2} < \varepsilon \iff \frac{n+3n}{n^2} < \varepsilon \Rightarrow \frac{n+3}{n^2} < \varepsilon \Rightarrow$$

$$\frac{|n-3|}{n^2} < \varepsilon \Rightarrow \left|\frac{n-3}{n^2}\right| < \varepsilon \Rightarrow \left|\frac{n-3}{n^2+9}\right| < \varepsilon \iff \left|\frac{n-3}{n^2+9} - 0\right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this gives the result.

(d) Assume, for the sake of contradiction, that s_n converges to some $s \in \mathbb{R}$. Then, for $\varepsilon = 1$, there exists $N \in \mathbb{R}$ so that $n > N$ ensures

$$|s_n - s| < 1 \iff s-1 < s_n < s+1$$
$$\iff s-1 < (n+1)^2 - 2 < s+1$$
$$\iff s+1 < (n+1)^2 < s+3$$

By the lemma following the Archimedean Property, $\exists\ m \in \mathbb{N}$ so that $m > s+3$. Let $k = \max(m, N+1)$. Then $\underline{k \geq m > s+3}$ ✪ and $k > N$. The latter ensures:

$$(k+1)^2 < s+3 \Rightarrow k < k^2 + 2k + 1 < s+3$$

This contradicts ✪. Thus s_n must not converge to any $s \in \mathbb{R}$.

3

a

Pf: For all $n \in \mathbb{N}$, $\sup(A) - \frac{1}{n} < \sup(A)$. Since $\sup(A)$ is the *least* upper bound of A, $\sup(A) - \frac{1}{n}$ cannot be an upper bound for A. That is, $\exists\ a_n \in A$ s.t. $\sup(A) - \frac{1}{n} \leq a_n$. Since $a_n \in A$, for all $n \in \mathbb{N}$, we have

$$\sup A - \frac{1}{n} \leq a_n \leq \sup A.$$

(b)

Since $\sup A - \frac{1}{n} \le a_n \le \sup A$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \sup A - \frac{1}{n} = \lim_{n\to\infty} \sup A = \sup A$ by squeeze lemma we obtain $\lim a_n = \sup A$.

(c) Fix $n \in \mathbb{N}$. Since A is not bounded above, n is not an upper bound for A, so there exists $a_n \in A$ s.t. $a_n > n$. In this way, we know there exists a sequence a_n with $a_n \in A$ and $a_n > n$ for all $n \in \mathbb{N}$.

(d) Fix $m > 0$. Let $N = m$. Then for all $n > N$, $a_n > n > N = m$. Thus a_n diverges to $+\infty$. This shows $\lim_{n\to\infty} a_n = \sup A$

(4)

(1) (i) no, $\sup(C) = +\infty$

(ii) yes, $\inf(C) = 0$

(2) (d)

(3) d