Practice Midterm 2 Solutions

①

First, we show $\liminf_{n\to\infty} s_n$ is an upper bound for A. Fix $a \in A$. Then $\exists$ N_a s.t. $n \geq N_a$ ensures $s_n \geq a$. Thus, for $N > N_b$, $b_N := \inf\{s_n : n > N\} \geq a$. By the contrapositive of HW4, Q6ⓐ, this gives

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}$$
$$\geq a$$

Our argument above shows that $\liminf_{n\to\infty} s_n \geq \sup(A)$. Suppose, for the sake of contradiction, that $\sup(A) < \liminf_{n\to\infty} s_n$. Then $\exists\, r \in \mathbb{R}$ s.t $\sup(A) < r < \liminf_{n\to\infty} s_n$. Since $r \notin A$, $|\{n \in \mathbb{N} : s_n < r\}|$ is infinite. Thus, $\inf\{s_n : n > N\} \leq r$ $\forall\, N \in \mathbb{N}$.

This implies $\liminf_{n\to\infty} s_n \leq r$, which is a contradiction.

ⓑ $\sup \emptyset = -\infty$

(2)

(2)(a) We proceed by induction. For the base case, note that $s_1 = 1$, $s_2 = 2$, $s_3 = \frac{3}{2}$, $s_4 = \frac{5}{3}$. Suppose $s_{2k} \geq s_{2(k+1)}$ and $s_{2k-1} \leq s_{2k+1}$.
Then $s_{2(k+2)} = 1 + \frac{1}{s_{2k+1}} \leq 1 + \frac{1}{s_{2k-1}} = s_{2k}$,
so $s_{2k+3} = 1 + \frac{1}{s_{2(k+2)}} \geq 1 + \frac{1}{s_{2k}} = s_{2k+1}$.

(b) Since s_{2n} is decreasing and $s_2 = 2$, we have $s_{2n} \leq 2 \ \forall n$. Since s_{2n-1} is increasing and $s_1 = 1$, we have $s_{2n-1} \geq 1 \ \forall n$. Furthermore,

$$s_{2n} = 1 + \frac{1}{s_{2n-1}} \overset{s_{2n-1}>0}{\geq} 1 \ \forall n \text{ and}$$

$$s_{2n+1} = 1 + \frac{1}{s_{2n}} \overset{s_{2n}\geq 1}{\leq} 2.$$ This shows $1 \leq s_n \leq 2 \ \forall n$.

(c) Since $1 \leq s_n \leq 2$ for all n, the subsequence of even terms and the subsequence of odd terms are both bounded and monotone. Hence, they both converge. Let $\lim_{k\to\infty} s_{2k} = s_{even}$ and $\lim_{k\to\infty} s_{2k-1} = s_{odd}$.

Note that:

$$s_{2k+1} = 1 + \frac{1}{s_{2k}} = 1 + \frac{1}{1 + \frac{1}{s_{2k-1}}}.$$

Since $s_{2k-1} \geq 1$, $s_{odd} \geq 1$. Thus, applying the limit theorems (quotient, sum), we have $s_{odd} = \lim_{k \to \infty} s_{2k+1} = 1 + \frac{1}{1 + \frac{1}{s_{odd}}}$.

Thus, s_{odd} solves $(s_{odd} - 1) = \left(1 + \frac{1}{s_{odd}}\right)^{-1}$

$\overset{s_{odd} \geq 1}{\iff} (s_{odd} - 1)\left(1 + \frac{1}{s_{odd}}\right) = 1 \iff s_{odd} + 1 - 1 + \frac{1}{s_{odd}} = 1$

$\iff s_{odd}^2 - s_{odd} - 1 = 0$. By the quadratic formula and the fact that $s_{odd} \in [1,2]$, we obtain $s_{odd} = \varphi$.

Finally,

$$s_{2k} = 1 + \frac{1}{s_{2k-1}}.$$

Again, applying the limit theorems, we obtain $s_{even} = 1 + \frac{1}{s_{odd}} \Rightarrow s_{even} = \varphi$.

(d) Fix $\varepsilon > 0$. Choose N_{even} so $k > N_{even}$ ensures $|s_{2k} - \varphi| < \varepsilon$ and N_{odd} so $k > N_{odd}$ ensures $|s_{2k-1} - \varphi| < \varepsilon$.

Let $N = 2 \cdot \max\{N_{even}, N_{odd}\}$. Then

$n > N$ ensures that either $n = 2k$ and $k > N_{even}$ or $n = 2k-1$ and $k > N_{odd}$. In either case $|s_n - \varphi| < \varepsilon$.

③

a) $$\limsup_{n\to+\infty} s_n = \lim_{N\to+\infty} \sup\{s_n : n > N\}$$

b) Suppose $\limsup_{n\to+\infty} |s_n| = 0$. Since $|s_n| \geq 0$, $\liminf_{n\to+\infty} |s_n| = 0$. Thus $\lim_{n\to+\infty} |s_n| = 0$ and $\lim_{n\to+\infty} -|s_n| = 0$. Since $-|s_n| \leq s_n \leq |s_n|$, the squeeze lemma ensures $\lim_{n\to+\infty} s_n = 0$.

c) Suppose $\lim_{n\to+\infty} s_n = 0$. As shown on Midterm 1, $\lim_{n\to+\infty} |s_n| = |\lim_{n\to+\infty} s_n| = |0| = 0$. Thus $\limsup_{n\to+\infty} |s_n| = 0$.

4)

1.

(a)

(b)

2.

(i) True

(ii) False - consider s_n = (-1)^n

4

1. (a) $\limsup_{n\to\infty} a_n = 1$, $\liminf_{n\to\infty} a_n = -1$

   (b) No, No

2. (a) True

   (b) False