Practice Final Solutions

(1)

(a) Define a sequence s_n as follows: since M is the least upper bound for $\{f(x) : x \in S\}$, $\forall n \in \mathbb{N}$, $\exists x_n \in S$ s.t. $M - \frac{1}{n} < f(x_n) \leq M$. Thus, by the Squeeze Lemma, $\lim_{n\to\infty} f(x_n) = M$.

(b) Since $x_n \in S$ and S is bounded, x_n is a bounded sequence. By Bolzano-Weierstrass, it has a convergent subsequence x_{n_k}. Since $f(x_{n_k})$ is a subsequence of $f(x_n)$, $\lim_{k\to\infty} f(x_{n_k}) = M$. Thus, the result holds with $t_k = x_{n_k}$.

(2)

(a) Fix $a, b \in \mathbb{R}$ with $a < b$. Then we also have $\frac{a}{\sqrt{5}} < \frac{b}{\sqrt{5}}$. By density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists$ $r \in \mathbb{Q}$ s.t.

$$\frac{a}{\sqrt{5}} < r < \frac{b}{\sqrt{5}}.$$

Thus, $a < \sqrt{5}r < b$. Since $\sqrt{5}r \in S$, this shows the result.

(b) Fix $x \in \mathbb{R}$. By part (a), $\forall n \in \mathbb{N}$, $\exists s_n \in S$ s.t. $x < s_n < x + \frac{1}{n}$. Then $\lim_{n\to\infty} s_n = x$, by the Squeeze Lemma. Since f is continuous, $\lim_{n \to \infty} f(s_n) = f(x)$. Finally, since $f(s_n) = \pi \ \forall n \in \mathbb{N}$, we see that $f(x) = \pi$.

(c) Let $h(x) = f(x) - g(x) + \pi$. Since f and g are continuous and constant functions are continuous, using that the sum and product of continuous functions is continuous, we have

$$h(x) = \underbrace{f(x) + \overbrace{(-g(x))}^{\text{cts, since product of } g \text{ and } -1} + \pi}_{\text{sum of cts is cts}}$$

is continuous.

Furthermore, by definition, $h(s) = \pi$ for all $s \in S$. By part (a), we see $h(x) = \pi$ for all $x \in \mathbb{R}$. This shows $f(x) - g(x) = 0 \ \forall x \in \mathbb{R}$, thus $f(x) = g(x)$ for all $x \in \mathbb{R}$.

③ (i) False - suppose $x_n = y_n = (1, 1, 1, \dots)$.

Then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 1$, so

$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} y_n$, but $x_n = y_n$ for all $n \in \mathbb{N}$.

(ii) True. Assume, for the sake of contradiction that there exist infinitely many n so that $x_n \geq y_n$. Then, for any $N \in \mathbb{N}$, there exists $k > N$ so that $x_k \geq y_k$. Thus, $\forall N \in \mathbb{N}$,

$$\underbrace{\sup\{x_n : n > N\}}_{a_N} \geq x_k \geq y_k \geq \underbrace{\inf\{y_n : n > N\}}_{b_N}.$$

Since the limits of a_N and b_N exist, this shows

$\limsup_{n\to\infty} x_n = \lim_{N\to\infty} a_N \geq \lim_{N\to\infty} b_N = \liminf_{n\to\infty} y_n$.

This is a contradiction.

4. (a) Define $h(x) = f(x) - g(x)$. Since f and g are continuous and the constant function $\varphi(x) = -1$ is continuous,

$$h(x) = \underbrace{f(x) + \overbrace{(-g(x))}^{\text{product } \varphi g \text{ is continuous}}}_{\text{sum is continuous}},$$

so $h(x)$ is continuous. Then $h(a) \geq 0 \geq h(b)$, so by the IVT, $\exists\, x_0 \in [a,b]$ s.t. $h(x_0) = 0$. Thus $f(x_0) = g(x_0)$, which gives the result.

(b) Define $g(x) = x$. This function is continuous since $\forall\, x_0 \in \mathbb{R}$, $x_n \to x_0$

$$\lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} x_n = x_0 = g(x_0).$$

Since $f(0) \geq g(0)$ and $f(1) \leq g(1)$, by part (a), $\exists\, x_0 \in [0,1]$ s.t. $f(x_0) = g(x_0) = x_0$.

5)

(1) (a) False, consider the constant function $f(x) = 1.5$.

(b) False, consider $f(x) = x$. Then the infimum is zero, but there is no x_0 in $(0,1)$ for which $f(x_0) = 0$.

(2) (i) (d)

(ii) (e)

(iii) (e)

(iv) (b)