Math 117: Practice Final Exam

All questions are extra practice. The final exam will be five questions long. The format will be very similar to this practice exam, but the questions will be different. Don't forget that one of the extra practice problems from the homework will be on the exam :).

Question 1

Suppose $S \subseteq \mathbb{R}$ is a bounded set and f is a bounded function on S. Define $M = \sup\{f(s) : s \in S\}$.

(a) Use the definition of the supremum to prove that there exists a sequence s_n of elements in S so that $\lim_{n\to+\infty} f(s_n) = M$.

(b) Now, use the result from part (a) to prove that there exists a *convergent* sequence t_k of elements in S so that $\lim_{k\to+\infty} f(t_k) = M$.

Question 2

Consider the set $S = \left\{\sqrt{5}r : r \in \mathbb{Q}\right\}$. You may assume that $\sqrt{5}$ is an irrational number.

(a) Prove that S is dense in $\mathbb{R}$ by showing that, for all $a, b \in \mathbb{R}$ with $a < b$, there exists $s \in S$ satisfying $a < s < b$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that if $f(s) = \pi$ for all $s \in S$, then $f(x) = \pi$ for all $x \in \mathbb{R}$.

(c) Let f and g be continuous real-values functions defined on $\mathbb{R}$ such that $f(s) = g(s)$ for each $s \in S$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$. (Hint: define a new function and apply part (b).)

As a consequence, you see that if two continuous functions on $\mathbb{R}$ are equal on a dense subset of $\mathbb{R}$, they must actually be equal everywhere.

Question 3

Determine whether the following statements are true or false. If they are true, prove them. If they are false, provide a counterexample and justify your counterexample.

(i) If $\limsup_{n\to+\infty} x_n \leq \liminf_{n\to+\infty} y_n$, then $x_n \geq y_n$ for at most finitely many n.

(ii) If $\limsup_{n\to+\infty} x_n < \liminf_{n\to+\infty} y_n$, then $x_n \geq y_n$ for at most finitely many n.

Question 4

(a) Let f and g be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Use the intermediate value theorem to prove that $f(x_0) = g(x_0)$ for at least one x_0 in $[a, b]$.
(Hint: apply the IVT to $h(x) = f(x) - g(x)$.)

(b) Let f be a continuous function mapping $[0, 1]$ into $[0, 1]$, i.e. f is a function defined on the interval $[0, 1]$ so that $f(x) \in [0, 1]$ for all $x \in [0, 1]$. Use part (a) to show that there exists a point $x_0 \in [0, 1]$ so that $f(x_0) = x_0$. (Such a point is called a *fixed point* since f maps the point to itself, leaving its location "fixed".)
(Hint: Again, you want to cook up a new function and apply the IVT.)

Question 5: Lightning Round

1) State whether the following statements are true or false. If they are false, provide a counterexample. You do not need to justify your counterexample.

   (a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(0) > 0$ and $f(1) < 2$, then there exists $x_0 \in (0, 1)$ so that $f(x_0) = 1$.

   (b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, there exists $x_0 \in (0, 1)$ so that $f(x_0) = \inf\{f(x) : x \in (0, 1)\}$.

2) Select the best choice:

   (i) Fix $a \in \mathbb{R}$ and consider the function $f(x) = \begin{cases} x^2 \sin(1/x) & \text{for } x \neq 0, \\ a & \text{for } x = 0. \end{cases}$

   If ________________, then ________________.

   (a) $a \neq 0$, f is continuous
   (b) $a > 0$, f is continuous
   (c) $a < 0$, f is continuous
   (d) $a = 0$, f is continuous
   (e) $a = 0$, f is discontinuous

   (ii) Fill in the blanks: "A function f is continuous at $x_0 \in \text{dom}(f)$ if and only if ________________ s.t. $x \in \text{dom}(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \epsilon$.

   (a) $\forall \epsilon > 0$, $\exists \delta \geq 0$
   (b) $\exists \epsilon > 0$ s.t. $\forall \delta > 0$
   (c) $\forall \epsilon > 0$, $\forall \delta > 0$
   (d) $\exists \epsilon > 0$ s.t. $\exists \delta > 0$
   (e) $\forall \epsilon > 0$, $\exists \delta > 0$

   (iii) Given $f(x) = x$ and $g(x) = 1 + x^2$, which function is *not* continuous on $\mathbb{R}$?

   (a) $f \circ g(x)$
   (b) $f(x)g(x)$
   (c) $f(x)/g(x)$
   (d) $g \circ f(x)$
   (e) $g(x)/f(x)$

   (iv) Suppose f is a bounded function. Then ______ is a bounded set.

   (a) $\{f(x) : x \in \mathbb{R}\}$
   (b) $\{f(x) : x \in \text{dom}(f)\}$
   (c) $\{x : f(x) > 0\}$
   (d) $\{x : f(x) < 0\}$
   (e) $\{x : 1/f(x) > 0\}$