Math 117: Practice Midterm 1

All questions are extra practice. The midterm will be four questions long, where the last of the four questions is a "lightning round". The first two questions will be given on Tuesday, and the second two questions will be given on Thursday. The format will be very similar to this practice exam, but the questions will be different. Don't forget that one of the extra practice problems from HW1, HW2, HW3, or HW4 will be on the midterm :).

Question 1

Let $(0, +\infty)$ denote the set $\{x \in \mathbb{R} : x > 0\}$.

(a) State the definition of what it means for a nonempty set $S \subseteq \mathbb{R}$ to be bounded below.

(b) Consider a nonempty set $S \subseteq (0, +\infty)$. Define $S' = \{1/s : s \in S\}$. Prove that $a > 0$ is a lower bound for S if and only if $1/a > 0$ is an upper bound for S'.

(c) Suppose $\inf S > 0$. Prove that $\sup S' = 1/\inf S$.

(d) Suppose $\inf S = 0$. Prove that $\sup S' = +\infty$.

Question 2

(a) State the definition of a convergent sequence.

(b) State the definition of what it means for a sequence to *not* converge.

(c) Use the definition of a convergent sequence to prove that $\lim_{n\to+\infty} \frac{n-3}{n^2+9} = 0$.

(d) Use the definition of a convergent sequence to prove that the sequence $s_n = (n+1)^2 - 2$ does not converge.

Question 3

Consider a nonempty set $A \subseteq \mathbb{R}$.

(a) Suppose A is bounded above. Prove that there exists a sequence a_n, satisfying

$$\{a_n : n \in \mathbb{N}\} \subseteq A$$

and

$$\sup A - \frac{1}{n} \leq a_n \leq \sup A \text{ for all } n \in \mathbb{N}.$$

(Hint: Use the fact that $\sup A - \frac{1}{n}$ *cannot* be an upper bound.)

(b) Prove that the sequence you found in the previous part satisfies $\lim_{n\to\infty} a_n = \sup A$.

(c) Now suppose A is not bounded above. Prove that there exists a sequence a_n satisfying

$$\{a_n : n \in \mathbb{N}\} \subseteq A$$

and

$$a_n \geq n \text{ for all } n \in \mathbb{N}.$$

(d) Prove that the sequence you found in the previous part satisfies $\lim_{n\to+\infty} a_n = \sup A$

In summary, you have proved the following important result: for any nonempty set $A \subseteq \mathbb{R}$, we may always find a sequence of elements a_n in A so that $\lim_{n\to+\infty} a_n = \sup A$.

Question 4

Lightning Round!
You do not need to show your work or justify your answers.

(1) Consider the set $C = \left\{ n^2 + \frac{(-1)^n}{n^2} : n \in \mathbb{N} \right\}$.

(i) Does the set have a maximum? What is $\sup(C)$?

(ii) Does the set have a minimum? What is $\inf(C)$?

(2) Circle the correct answer:

Given a nonempty set $S \subseteq \mathbb{R}$, if M_0 is the supremum of S and M is a /an __________ , then $M_0 \leq M$.

(a) infimum of S
(b) minimum of S
(c) additive identity
(d) upper bound of S
(e) multiplicative identity

(3) Circle the correct answer:

Given $S \subseteq \mathbb{R}$ nonempty, what criteria must S satisfy in order to have a supremum?

(a) $\max(S)$ D.N.E.
(b) S has finitely many elements
(c) S is unbounded
(d) S is bounded above
(e) S is bounded below

