Math 117: Practice Midterm 2

All questions are extra practice. The midterm will be four questions long, where the last of the four questions is a "lightning round". The format will be very similar to this practice exam, but the questions will be different. Don't forget that one of the extra practice problems from HW5, HW6, or HW7 will be on the midterm :).

Question 1

(a) Let s_n be a bounded sequence of real numbers. Let A be the set of $a \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n < a\}$ is finite. In other words, A is the set of real numbers a for which at most finitely many s_n are less than a. Prove that $\sup A = \liminf s_n$.

(b) How would you want to define $\sup \emptyset$, where $\emptyset$ is the empty set, in order to make the result true for unbounded sequences s_n? You do not need to justify your answer.

Question 2

The golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ is the limit of the following continued fraction:

$$1 + \cfrac{1}{1 + \frac{1}{1+\frac{1}{1+\dots}}}$$

This problem will guide you through the steps to prove this. Define a sequence s_n recursively as follows: $s_1 = 1$, and, for $n \geq 1$, $s_{n+1} = 1 + \frac{1}{s_n}$.

(a) Prove that $1 \leq s_n \leq 2$ for all $n \in \mathbb{N}$. (Hint: first show $s_n \geq 0$.)

(b) Prove that the subsequence of even elements, s_{2n}, is decreasing and the subsequence of odd elements, s_{2n-1}, is increasing. (Hint: prove both facts simultaneously by induction.)

(c) Prove that the subsequence of even elements and the subsequence of odd elements converge to $\varphi = \frac{1+\sqrt{5}}{2}$.

(d) Use part (c) to prove that the entire sequence s_n converges to $\varphi = \frac{1+\sqrt{5}}{2}$.

Question 3

(a) State the definition of $\limsup s_n$.

(b) Suppose $\limsup |s_n| = 0$. Prove that $\lim s_n = 0$.

(c) Suppose $\lim s_n = 0$. Prove that $\limsup |s_n| = 0$.

Question 4 (17 points)

Lightning Round!
You do not need to show your work or justify your answers.

(1) Circle the correct answer:

If s_n is *not* bounded and *is* monotone, what *must always* be true about s_n?

(a) limit exists

(b) convergent

(c) decreasing

(d) increasing

(e) Cauchy

Suppose s_n is bounded. Which property of $b_N = \inf\{s_n : n > N\}$ is *not* always true?

(a) convergent

(b) decreasing

(c) monotone

(d) bounded below

(e) Cauchy

(2) True or False? If false, write a counterexample. You do not need to justify your counterexample.

(i) Suppose s_n is **not** a bounded sequence. Then either $\limsup s_n = +\infty$ or $\liminf s_n = -\infty$ or both.

(ii) Suppose s_n is a bounded sequence. Then $\limsup s_n = \liminf s_n$.

