Lecture 10

Announcements:
* On Monday, Oct 31st office hours will be 12-1pm.

Recall:

Integration of nonnegative measurable fns

$$(X, \mathcal{M}, \mu)$$ measure space

Def: Given $f : X \rightarrow [0, +\infty]$ measurable, \\
$Sf \mu = \sup \{ \int \varphi d\mu : 0 \leq \varphi \leq f, \varphi \text{ simple} \}$

Rmk:

* For $c \geq 0$, $S(cf) \mu = cSf \mu$
* $f \leq g$ nonneg, meas $\Rightarrow Sf \mu \leq Sg \mu$
MAJOR THEOREM #2

**Thm**: (Monotone Convergence Theorem): Given $\{f_n\}_{n=1}^\infty$ nonnegative measurable functions s.t. $f_n \leq f_{n+1}$ $\forall n \in \mathbb{N}$, then

$$\lim_{n \to \infty} \int f_n \, d\mu = \int \lim_{n \to \infty} f_n \, d\mu.$$ 

**Thm**: (Beppo-Levi): Given $\{f_n\}_{n=1}^\infty$ nonnegative, measurable functions,

$$\sum_{n=1}^\infty \int f_n \, d\mu = \int \sum_{n=1}^\infty f_n \, d\mu.$$

**Proof**:

Last time, we showed that $\forall N \in \mathbb{N}$,

$$\sum_{n=1}^N f_n \leq \sum_{n=1}^\infty f_n.$$

Thus, by the monotone convergence thm,

$$\sum_{n=1}^\infty f_n = \lim_{N \to \infty} \sum_{n=1}^N f_n = \lim_{N \to \infty} \sum_{n=1}^\infty f_n \leq \sum_{n=1}^\infty f_n.$$

Therefore,

$$\sum_{n=1}^\infty f_n = \int \lim_{N \to \infty} \sum_{n=1}^N f_n = \int \sum_{n=1}^\infty f_n.$$

$\square$
Without monotonicity of the sequence \( fn \), you can still get an inequality for limitings.

**Thm (Fatou's Lemma):** Given \( \{fn\}_{n=1}^{\infty} \) nonnegative, measurable

\[
\lim_{n \to \infty} \int fn \, d\mu \geq \int \liminf_{n \to \infty} fn \, d\mu.
\]

**Proof:**

By definition, \( \liminf_{n \to \infty} fn = \lim_{n \to \infty} \inf_{k \geq n} fn \).

Furthermore, \( gn \leq gn+1 \quad \forall n \in \mathbb{N} \).

Thus, by MCT,

\[
\lim_{n \to \infty} \int gn \, d\mu = \int \lim_{n \to \infty} gn \, d\mu = \int \liminf_{n \to \infty} fn \, d\mu.
\]

By defn., \( gn \leq fn \), so \( \int gn \leq \int fn \).

Taking \( \lim_{n \to \infty} \) of both sides,

\[
\lim_{n \to \infty} \int fn \geq \lim_{n \to \infty} \int gn = \lim_{n \to \infty} \int gn.
\]

This gives the result. \( \square \)
Two important examples where strict inequality holds:

\[(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{M}_\infty, \lambda)\]

Ex: (“run away to infinity”)

\[f_n = 1_{[n, n+1]} \Rightarrow \lim_{n \to \infty} f_n = 0\]

\[
\begin{array}{c}
\downarrow \\
\lim_{n \to \infty} \int f_n \, d\lambda = 1 > 0 = \lim_{n \to \infty} \int \lambda \\
\end{array}
\]

Ex: (“goes up the spout example”)

\[f_n = n 1_{[0, \frac{1}{n}]} \Rightarrow \lim_{n \to \infty} f_n = \begin{cases} 0 & \text{for } x \neq 0 \\ +\infty & \text{for } x = 0 \end{cases}\]

\[
\begin{array}{c}
\downarrow \\
\end{array}
\]
\[
\lim_{n \to 0} S_{fndx} = 1
\]
\[
S_{fndx} = \sum_{n=0}^{\infty} g_{n} \lambda d\lambda
\]
\[
= \lim_{n \to \infty} S_{g_{n} d\lambda}
\]
\[
= \lim_{n \to \infty} 0 \cdot \lambda(R \setminus E_{0}) + n \lambda(E_{0})
\]
\[
= 0.
\]

Let \( g_{n}(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ n & \text{if } x = 0 \end{cases} \)
\( g_{n} \leq g_{n+1} \)
\[
\lim_{n \to \infty} g_{n} = \lim_{n \to \infty} f_{n}
\]

Prop: Given \( f : X \to [0, +\infty] \) measurable, if \( \int f d\mu < +\infty \),
(i) \( \{ x : f(x) = +\infty \} \) is an null set
(ii) \( \{ x : f(x) > 0 \} \) is \( \sigma \)-finite

Pg: HW6
Prop: Given \( f: X \to [0,+\infty] \) measurable,
\[
S\delta f \mu = 0 \iff f = 0 \text{ \( \mu \)-a.e.}
\]

Proof: First, suppose \( f \) is simple,
\[
f = \sum_{i=1}^{n} a_i 1_{E_i}.
\]
Then,
\[
S\delta f \mu = 0 \iff \sum_{i=1}^{n} a_i \mu(E_i) = 0
\]
\[
\iff \text{either } a_i = 0 \text{ or } \mu(E_i) = 0 \text{ for all } i = 1, \ldots, n
\]
\[
\iff f = 0 \text{ a.e.}
\]
For more general \( f \),
Suppose \( f = 0 \text{ \( \mu \)-a.e.} \). Then,
\[
(\forall) \quad S\delta f \mu = \sup \{ S\delta \delta f \mu : 0 \leq \delta \leq f, \delta \text{ simple} \}
\]
\[
= 0
\]
Conversely, if \( \text{St} \delta \mu = 0 \), then by defn \((\star)\), we must have

\[ \int \delta \phi \, d \mu = 0, \quad \forall \phi \text{ simple}, \ 0 \leq \phi \leq f. \]

Assume, for the sake of contradiction, that \( f = 0 \) \( \mu \)-a.e. fails, that is, \( \exists \) a set \( A \) with \( \mu (A) > 0 \) so that \( f(x) > 0 \) for all \( x \in A \).

Note that \( \{ x : f(x) > 0 \} = \bigcup_{i=1}^{\infty} \{ x : f(x) > \frac{1}{n^2} \} \). Then

\[ \mu (A) \leq \mu (\{ x : f(x) > 0 \}) \leq \sum_{n=1}^{\infty} \mu (E_n). \]

Thus \( \mu (E_n) > 0 \) for some \( n \in \mathbb{N} \).

Let \( \phi = \frac{1}{n} 1_{E_n} \). Then \( 0 \leq \phi \leq f \).

But

\[ \int \phi \, d \mu = \frac{1}{n} \mu (E_n) > 0. \]

This contradicts \( (\star\star) \). \( \square \)
Integration of Real Functions

Measure space \((X, M, \mu)\)

Given \(f: X \to \mathbb{R}\),

- "positive part"
  \[ f^+ = f \vee 0 \]
  \[ f^- = (-f) \vee 0 \]
  \[ f = f^+ - f^- \]

- "negative"

Def: Given \(f: \mathbb{R} \to \mathbb{R}\) measurable, if either \(\int f^+\) or \(\int f^-\) is finite,

\[ \int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu \]

If both \(\int f^+ \, d\mu\) and \(\int f^- \, d\mu\) are finite, \(f\) is integrable and we write \(f \in L^1(\mu)\).

Prop: \(L^1(\mu)\) is a real vector space and \(\int \cdot \, d\mu\) is a linear functional on it.
Pg: Fix \( f, g \in L^2(\mu), a, b \in \mathbb{R} \).

Note \( af + bg \) is measurable and 

\[
|af + bg| \leq |a||f| + |b||g|,
\]

so 

\[
|af + bg| = \sqrt{a^2|f|^2 + b^2|g|^2}.
\]

Thus \( af + bg \in L^2(\mu) \).

Let \( f \in L^2(\mu), a \geq 0, \)

\[
|af| = |a||f| = af,
\]

For \( a \leq 0 \), the result follows by replacing \( f \) with \(-f\).

Thus, \( \forall f \in L^2(\mu), a \in \mathbb{R}, Sa f = af \).
Finally, \( f, g \in L^2(\mu) \),

\[
\begin{align*}
S(f+g) &= S(f)+S(g) - \left( S(f^-) + S(g^-) \right) \\
&= Sf + Sg \\
&\leq |Sf^+| + |Sf^-| \\
&= Sf^+ + Sf^- = |Sf|.
\end{align*}
\]

Prop: If \( f \in L^2(\mu) \), then

\[|Sf| = |Sf^+| - |Sf^-| \leq |Sf^+| + |Sf^-| = Sf^+ + Sf^- = |Sf|.\]

Note that

\[\begin{align*}
(f+g)_+ - (f+g)_- &= f_+ - f_- + g_+ - g_- \\
(f+g)_+ + f_- + g_- &= f_+ + g_+ + (f+g)_- \\
S(f+g)_+ + Sf_+ + Sg_- &= Sf_+ + Sg_- + S(f+g)_-
\end{align*}\]
Prop: If \( f, g \in L^2(\mu) \), then
\[ \int |f - g| \, d\mu = 0 \iff f = g \mu\text{-a.e.} \]

Proof: This is an immediate corollary of previous prop about nonnegative measurable functions, taking \( h = |f - g| \).
\[ \|f - g\|_1 \leq \int |f - g| \, d\mu \]

Moral: if you modify a function on a null set, it does not change the integral.

Consequently, even if a function \( f \) is only defined almost everywhere, \( \|f\|_1 \) is still well-defined, since we may take \( f \) to be equal to any element of \( \mathbb{R} \) (e.g. \( 0 \)) where it is not defined, and it won't affect \( \|f\|_1 \).

We have already shown \( L^1(\mu) \) is a vector space... Is it a metric space?
\[ d_{L^1}(f,g) = \int |f-g| \, d\mu \]

\[ f = 1_{\mathbb{E}^3}, \quad g = 1_{\mathbb{E}^3}, \quad d_{L^1}(f,g) = 0. \]

We see that, when \( d_{L^1} \) is defined on all integrable functions, it fails to be nondegenerate.

Solution...

**Def:**

\[ L^1(\mu) := \{ f : X \to \mathbb{R} \text{ measurable}, \int f \, d\mu < \infty \} \]

Where \( f \sim g \) iff \( f = g \) \( \mu \text{-a.e.} \).

**Remark:** By abuse of notation, let \( f \in L^1(\mu) \) denote...

- the equivalence class
- a representative of that equivalence class
- a representative that is only defined \( \mu \text{-a.e.} \).
Prop: $\|f\|_{L^1(\mu)} = \int f d\mu$ is a norm on $L^1(\mu)$.

Pf: Triangle inequality $\checkmark$

Absolute homogeneity $\checkmark$

Nondegenerate $\checkmark$


Why do we need nonnegativity in Fatou's Lemma?

Ex: ("goes down the spout")

$(X, M, \mu) = (\mathbb{R}, M_2, \lambda)$

$f_n = -n \cdot 1_{[0, 1/n]}$

$n \to \infty f_n = \begin{cases} 0 & \text{if } x \neq 0 \\ -\infty & \text{if } x = 0 \end{cases}$

$\lim_{n \to \infty} s f_n = -1 \neq 0 = \lim_{n \to \infty} s f_n$. 


Is there a theorem that will allow us to interchange the limit and integral for real valued functions?

Dominated Convergence Theorem
Modes of Convergence:

$(X, c\mathcal{M}, \mu)$

$f_n, f : X \to \overline{\mathbb{R}}$