Let $a_n$ and $b_n$ be the sequences of all rational numbers so that $(a_n, b_n) \subseteq U$. Then $\bigcap_{n=1}^{\infty} (a_n, b_n) \subseteq U$.

To see the opposite containment, fix $u \in U$. Since $U$ is open, $\exists \varepsilon > 0$ s.t. $(u-\varepsilon, u+\varepsilon) \subseteq U$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $a_n, b_n$ so that $u \in (a_n, b_n) \subseteq (u-\varepsilon, u+\varepsilon)$. Thus $U \subseteq \bigcap_{n=1}^{\infty} (a_n, b_n)$.

Since $(a,b) = \bigcap_{n=1}^{\infty} [a,b+\frac{1}{n})$, it is clear that $E_3 \subseteq B_{\mathbb{R}}$, so $\mathcal{M}(E_3) \subseteq B_{\mathbb{R}}$. OTOH $(a,b) = \bigcup_{n=1}^{\infty} [a,b-\frac{1}{n}]$, so $(a,b) \subseteq \mathcal{M}(E_3)$ and (*) ensured $B_{\mathbb{R}} \subseteq \mathcal{M}(E_3)$.\fi
\[ \text{Since } (a, b] = (a, +\infty) \setminus (b, +\infty), \ M(E_5) \leq M(E_5) \]
so part (i) ensures \( B_{IR} = \mathcal{M}(E_3) \leq M(E_5) \)
Since \( (a, +\infty) = \bigcup_{n=1}^{\infty} (a, a+n) \), \( E_5 \leq B_{IR} \), so \( M(E_5) \leq B_{IR} \).

\[ \text{Since } [a, +\infty) = \bigcap_{n=1}^{\infty} (a-n, +\infty), \ \text{part (ii) ensures } \]
\( E_\eta \leq M(E_5) = B_{IR} \), so \( M(E_6) \leq B_{IR} \).

Since \( (a, +\infty) = \bigcup_{n=1}^{\infty} [a+n, +\infty) \leq M(E_\eta) \), part (ii)
enhanced \( B_{IR} \leq M(E_6) \).

5) By defn, \( C^\mathcal{A} \) is nonempty.
First, we show closure under complements. 
Suppose \( E \in C^\mathcal{A} \). Then either \( E \) or \( E^c \)
is at most countably infinite \( \iff \)
either \( E^c \) or \( (E^c)^c \) is at most countably infinite \( \iff \)
\( E^c \in C^\mathcal{A} \).
Next, we show closure under countable unions. Suppose $E_1, E_2, \ldots \in \mathcal{A}$. Consider $\bigcup_{n=1}^\infty E_n$. If $\bigcup_{n=1}^\infty E_n$ is at most countably infinite, $\bigcap_{n=1}^\infty E_n \in \mathcal{A}$. On the other hand, suppose $\bigcup_{n=1}^\infty E_n$ is not at most countably infinite. Then there exists at least one set $E_m$ in the sequence that is not countable. By defn of $\mathcal{A}$, $E_m$ is at most countably infinite. Thus, $\left( \bigcup_{n=1}^\infty E_n \right)^c = \bigcap_{n=1}^\infty E_n^c \subseteq E_m^c$ is at most countably infinite, so $\bigcup_{n=1}^\infty E_n \in \mathcal{A}$.

First, we show $\limsup E_i = A_2$. Note that $x \in A_2 \iff x \in E_i$ for infinitely many $i$.

$\iff \forall k \in \mathbb{N}, \exists i \geq k \text{ s.t. } x \in E_i$

$\iff \forall k \in \mathbb{N}, x \in \bigcap_{i=k}^\infty E_i$

$\iff x \in \bigcap_{k=1}^\infty \bigcup_{i=k}^\infty E_i$

Next, we show $\liminf E_i = A_2$. Note that $x \in A_1 \iff x \in E_i$ for all but finitely many $i$.

$\iff \exists k \in \mathbb{N} \text{ s.t. } \forall i \geq k, x \notin E_i$

$\iff \exists k \in \mathbb{N} \text{ s.t. } x \in \bigcap_{i=k}^\infty E_i$

$\iff x \in \bigcap_{k=1}^\infty \bigcup_{i=k}^\infty E_i$
We must show it is closed under complements and countable unions. If \( E \in \mathcal{A} \), then
\[
F \setminus (E \cap F) = F \setminus (E \cap F) = F \setminus (E^c \cup F) = F \setminus E^c,
\]
which belongs to the \( \sigma \)-algebra, since \( E \in \mathcal{A} \).

If \( \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A} \), then
\[
\bigcup_{i=1}^{\infty} (E \cap F) = \left( \bigcup_{i=1}^{\infty} E_i \right) \cap F,
\]
which belongs to the \( \sigma \)-algebra, since \( \bigcup_{i=1}^{\infty} E_i \in \mathcal{A} \).

If both terms on the RHS were finite, \(|\mathcal{A}|\) would be finite. Thus at least one must be infinite.
(c)

**BASE CASE:**
- \( \exists F \in \mathcal{A} \) s.t. \( F \neq \emptyset, X \)
- Either \( \left| \{ E \in \mathcal{A}^2 \} \right| \) or \( \left| \{ E \in \mathcal{A}^2 \} \right| \) is infinite. If \( \text{1} \) is infinite, define \( Z_1 = F \). Otherwise, define \( Z_1 = F \).

**INDUCTIVE STEP:**
Consider a sequence \( Z_n \) of \( n \) disjoint, nonempty subsets belonging to \( \mathcal{A} \) s.t.
\[
\left| \bigcap_{E \in \mathcal{A}^2} E \right| = +\infty
\]

We will now show there exists \( Z_{n+1} \in \mathcal{A} \) s.t. \( Z_{n+1} \) is disjoint from \( Z_i \) for \( 1 \leq i \leq n \), \( Z_{n+1} \neq \emptyset \) and
\[
\left| \bigcap_{E \in \mathcal{A}^2} E \right| = +\infty
\]

Since \( \left| \bigcap \mathcal{A} \right| = +\infty \),
- \( \exists Y_{n+1} \) of the form \( Y_{n+1} = \bigcap_{j=1}^n \left( \bigcup_{j=1}^n \mathcal{Z}_j \right) \), satisfying \( Y_{n+1} \neq \left( \bigcap_{j=1}^n \left( \bigcup_{j=1}^n \mathcal{Z}_j \right) \right) \) and \( Y_{n+1} \neq \left( \bigcap_{j=1}^n \left( \bigcup_{j=1}^n \mathcal{Z}_j \right) \right) \).

In particular, note that \( Y_{n+1} \neq \left( \bigcup_{j=1}^n \mathcal{Z}_j \right) \).
One of the following is infinite

\[ \{ E \cap \left( \bigcup_{j=1}^{n} Z_j \right)^c \cap Y_{n+1} : E \in \mathcal{A} \} \]

\[ \{ E \cap \left( \bigcup_{j=1}^{n} Z_j \right)^c \cap \left( \bigcup_{j=1}^{n} Z_j \right)^c \cap Y_{n+1} : E \in \mathcal{A} \} \]

If the former is infinite, define

\[ Z_{n+1} = \left( \bigcup_{j=1}^{n} Z_j \right)^c \setminus Y_{n+1} = \left( \bigcup_{j=1}^{n} Z_j \right)^c \cap Y_{n+1}^c \]

Then,

\[ \left( \bigcup_{j=1}^{n+1} Z_j \right)^c = \left( \bigcup_{j=1}^{n} Z_j \right)^c \cap Z_{n+1}^c = \bigcup_{j=1}^{n} Z_j^c \cap Y_{n+1} \]

Thus,

\[ + \omega = \{ E \cap \left( \bigcup_{j=1}^{n} Z_j \right)^c \cap Y_{n+1} : E \in \mathcal{A} \} = |\mathcal{F}_{n+1}|. \]

Otherwise, if the latter is infinite, let \( Z_{n+1} = Y_{n+1} \). Then,

\[ \left( \bigcup_{j=1}^{n+1} Z_j \right)^c = \left( \bigcup_{j=1}^{n} Z_j \right)^c \cap Y_{n+1}^c \]

Thus

\[ + \omega = \{ E \cap \left( \bigcup_{j=1}^{n} Z_j \right)^c \cap \left( \bigcup_{j=1}^{n} Z_j \right)^c \cap Y_{n+1} : E \in \mathcal{A} \} = |\mathcal{F}_{n+1}|. \]
Finally note that $A$ ensures that $Z_{n+1}\cap (\bigcup_{j=1}^{n} Z_j) = \emptyset$ for either choice of $Z_{n+1}$.

(2) To see that $CA$ is uncountable, we will construct an injective map from $2^{\mathbb{N}} \to CA$. If $Z_n$ is the sequence defined in part (2), define $\phi(A) = \bigcup_{j \in A} Z_j$.

Since $Z_j$ is a sequence of disjoint sets, $\phi(A) = \phi(B) \Rightarrow \bigcup_{i \in A} Z_i = \bigcup_{j \in B} Z_j \Rightarrow A = B$, so $\phi$ is injective. Therefore, $|CA| \geq 2^{\mathbb{N}^1}$, so $CA$ is uncountable.
Suppose $f$ is continuous at $x \in \mathbb{R}$. Then, for all sequences \( x_n \to x \), \( \lim_{n \to \infty} f(x_n) = f(x) \).

By HW1, Q1, $f^*_x(x) = \inf \{ \lim f(x_n) : x_n \to x \}
\leq \inf \{ \limsup f(x_n) : x_n \to x \}
\leq \sup \{ \limsup f(x_n) : x_n \to x \}
= f_x(x)
= f^*_x(x)$

Similarly, $f^*_x(x) = \sup \{ \liminf f(x_n) : x_n \to x \}
= \sup \{ \lim f(x_n) : x_n \to x \}
\leq \inf \{ \liminf f(x_n) : x_n \to x \}
= f_x(x)
= f^*_x(x)$

Thus, equality holds throughout and, for all \( x_n \to x \),
\[ \limsup f(x_n) = f(x) \] Similarly, for all \( x_n \to x \),
\[ \lim f(x_n) = f(x) \] Therefore, by HW1, Q1,\[ \lim_{n \to \infty} f(x_n) = f(x) \] This shows \( f \) is continuous at \( x \).
b) It suffices to show that $E^c$, the set of points at which $f$ is continuous, is a countable intersection of open sets.

Thank you to Merrick for pointing out this solution only works for $f$ bounded (I changed the question from last year but not the solution!)

Recall that we always have $f^*(x) \leq f(x) \leq f^*(x)$. Therefore $x \in E^c \iff f^*(x) > f^*(x)$.

Consequently,

$$E^c = \{ x : f^*(x) > f^*(x) \} = \bigcap_{n \in \mathbb{N}} \{ x : f^*(x) - f^*(x) > -\frac{1}{n} \}.$$

Since $f^*$ and $-f^*$ are lower semicontinuous, $\bigcap_{n \in \mathbb{N}} \{ x : f^*(x) - f^*(x) > -\frac{1}{n} \}$ is open for extended real valued functions, this is not always well defined, since you can have $\infty - \infty$.

This shows $E$ can be written as the countable intersection of open sets.

Alternative (thanks to Merrick's friend Ben!)

$E = \{ x : f^*(x) < f^*(x) \}$

$$= \bigcup_{p,q \in \mathbb{R}, p < q} \{ x : f^*(x) \leq p \} \cup \{ x : q \leq f^*(x) \}$$

since $f^*$ is lsc, this is closed

$$= \bigcup_{p,q \in \mathbb{R}, p < q} f^* \left( (-\infty, p] \right) \cup \left( f^* \right)^{-1} \left( [q, \infty) \right)$$

since $f^*$ is usc, this is open.
This is a countable union of closed sets.