Lecture 10

Recall: *MAJOR THM #5*

**Thm (Monotone Convergence Theorem):** Given \( \{f_n \}_{n=1}^{\infty} : X \to [0, \infty] \) measurable functions s.t. \( f_n \leq f_{n+1} \) \( \forall n \in \mathbb{N} \), then

\[
\lim_{n \to \infty} \int f_n \, d\mu = \int \lim_{n \to \infty} f_n \, d\mu
\]

**Thm (Beppo-Levi):** Given \( \{f_n \}_{n=1}^{\infty} : X \to [0, \infty] \) measurable functions,

\[
\sum_{n=1}^{\infty} \int f_n \, d\mu = \int \sum_{n=1}^{\infty} f_n \, d\mu.
\]
**Thm (Fatou's Lemma):** Given $f_n : X \to [0, +\infty]$ measurable,

$$\liminf_{n \to \infty} \int f_n \, d\mu \geq \int \liminf_{n \to \infty} f_n \, d\mu.$$

**Ex:** ("runaway to infinity")

$$f_n = 1_{[n, n+1]}$$

**Ex:** ("goes up the spout")

$$f_n := n \cdot 1_{[0, \frac{1}{n}]}$$

**Prop:** Given $f : X \to [0, +\infty]$ measurable, if $\int f \, d\mu < +\infty$,

(i) $\{ x : f(x) = +\infty \}$ is a null set

(ii) $\{ x : f(x) > 0 \}$ is $\sigma$-finite.
Prop: Given \( f: X \to [0, +\infty) \) measurable,

\[
\int f \, d\mu = 0 \iff f = 0 \ \mu\text{-a.e.}
\]

\[
\mu(\{x: f(x) \neq 0^\pm\}) = 0
\]

Integration of Real Functions

Measure space \((X, \mathcal{E}, \mu)\).

Given \( f: X \to \mathbb{R} \), define

- "positive part": \( f_+ = f \vee 0 \)
- "negative part": \( f_- = (-f) \vee 0 \)

Then \( f = f_+ - f_- \) and \( |f| = f_+ + f_- \).
Def: Given $f: X \to \overline{\mathbb{R}}$, if either $\int f^+ \, d\mu$ or $\int f^- \, d\mu$ is finite,

$$\int f \, d\mu := \int f^+ \, d\mu - \int f^- \, d\mu$$

If both $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ are finite, we say $f$ is integrable and write $f \in L^1(\mu)$.

Prop: $L^1(\mu)$ is a real vector space and $f \mapsto \int f \, d\mu$ is a linear functional on $L^1(\mu)$. 
Last time, we showed that, if \( f,g \in L^1(\mu) \) and \( a,b \in \mathbb{R} \), then \( af + bg \in L^1(\mu) \). Thus \( L^1(\mu) \) is a real vector space.

Fix \( f \in L^1(\mu) \), \( a \geq 0 \), \( a \in \mathbb{R} \). Then both \( Saf^- = aSf^- \) and \( Saf^+ = aSf^+ \) are finite, so
\[
Saf = Saf^+ - Saf^- = aSf^+ - aSf^- = aSf
\]

If \( a < 0 \), the result follows by replacing \( f \) with \( -f \) above.

Finally, fix \( f,g \in L^1(\mu) \).
\[ Sf + g = S(f+g)^+ - S(f+g)^- \]
\[ = Sf^+ + Sg^+ - Sf^- - Sg^- \]
\[ = Sf + Sg \]

Note that

\[ (f+g)^+ - (f+g)^- = f^+ - f^- + g^+ - g^- \]

\[ (f+g)^+ + f^- + g^- = f^+ + g^+ + (f+g)^- \]

\[ S(f+g)^+ + Sf^- + Sg^- = Sf^+ + Sg^+ + S(f+g)^- \]

**Prop.** If \( f \in L^2(\mu) \), then

\[ |Sfd\mu| \leq S|fd\mu| \]
\[ |Sf| = |Sf_+ - Sf_-| \leq |Sf_+| + |Sf_-| \]

\[ = Sf_+ + Sf_- = |Sf| \]

Goal: Want to show \( L^2(\mu) \) is a metric space (in fact, a normed vector space).

Guess for metric:

\[ \|f-g\|_{L^2(\mu)} = \int |f-g|d\mu \]

Problem: \( f = 1_{\mathbb{E}} \), \( g = 0 \)

\[ \|f-g\|_{L^2(\mu)} = \int f d\mu = 0 \]
More generally, we have:

**Cor.** If $f, g \in L^1(\mu)$, then

$$\int |f-g|d\mu = 0 \iff f = g \text{ } \mu\text{-a.e.}$$

**Proof:** By previous prop...  

**Moral**

- If you modify an integrable function on a null set, it doesn't change the integral:

  $$|\int f d\mu - \int g d\mu| \leq \int |f-g| d\mu$$

- Even if a function $f$ is only defined $\mu$-a.e., $\int f d\mu$ is still uniquely determined.
Modified Def:

$L^2(\mu) := \{ f: X \to \mathbb{R} \text{ measurable}, \sigma f d\mu < +\infty \}$

where $f \equiv g$ iff $f = g$ $\mu$-a.e.

Remark: By abuse of notation, let $f \in L^2(\mu)$ denote...

1. the equiv class
2. a representative of the equiv class
3. a representative only defined $\mu$-a.e.

Prop: $\| f \|_{L^2(\mu)} := \sqrt{\int |f|^2 d\mu}$ is a norm on $L^2(\mu)$.

Pf: follows by previous propositions
Example: Why do we need hypothesis of nonnegativity in Fatou's Lemma?
Ex ("goes down the spout")

\[ f_n = -n \chi_{[0, \frac{1}{n}]} \]
\[ \lim_{n \to \infty} f_n(x) = \begin{cases} 
0 & \text{if } x > 0 \\
-\infty & \text{if } x < 0 
\end{cases} \]

\[ \lim_{n \to \infty} \int f_n \, d\lambda = 0 \geq -1 = \lim_{n \to \infty} \int f_n \, d\lambda. \]

Now, we use boundedness to come up with a theorem for interchanging limit + integral for real-valued fields.
Thm (Dominated Convergence Thm)

Given $\{f_n\}_{n=1}^{\infty} \subseteq L^1(\mu)$ s.t. $\lim_{n \to \infty} f_n$ exists $\mu$-a.e., if there exists $g \in L^1(\mu)$ s.t. $|f_n| \leq g$ $\mu$-a.e. $\forall n \in \mathbb{N}$, then

$\lim_{n \to \infty} \int f_n \, d\mu = \int \lim_{n \to \infty} f_n \, d\mu$

Rmk: $g$ does not have to be bounded

$(\mathbb{R}, \mathcal{M}^{\infty}, \lambda)$

$g(x) = \frac{1}{\lambda} 1_{[1,1]}(x)$

Rmk: $g$ bounded does not imply integrable.
\[ \text{If:} \]
\[ \text{Since } \lim_{n \to \infty} f_n \text{ exists } \mu\text{-a.e. and } |f_n| \leq g \text{ } \mu\text{-a.e. } \forall n \in \mathbb{N}, \text{ we have} \]
\[ \lim_{n \to \infty} |f_n| \leq g \text{ } \mu\text{-a.e.} \]

So \( \lim_{n \to \infty} f_n \in L^1(\mu) \).

Since \( g - f_n \geq 0 \) and \( g + f_n \geq 0 \) \( \mu \text{-a.e.} \), Fatou's Lemma ensures

\[ \lim_{n \to \infty} \int f_n \leq \int \lim_{n \to \infty} f_n \]

\[ \int g + \lim_{n \to \infty} \int f_n = \lim_{n \to \infty} \int g + f_n \geq \lim_{n \to \infty} \int g + f_n \]

\[ \implies \int g + \lim_{n \to \infty} f_n = \int g + \lim_{n \to \infty} f_n \]
Likewise, 
\[ S_g \limsup_{n \to \infty} S_{f_n} = \lim_{n} S_g - f_n \geq \lim_{n} g - f_n \geq \lim_{n} S_{f_n} = S_g - \lim_{n} f_n \]

Subtracting \( S_g \) from both sides and combining,

\[ \lim_{n} S_{f_n} \geq \lim_{n} f_n \geq \lim_{n} S_{f_n} \]

Thus, equality holds throughout \( D \).

We will now use DCT to identify two useful subsets of functions that are dense in \( L^1(\mu) \).
MAJOR THM 7

Thm: For any measure space \((X, \mathcal{M}, \mu)\), simple functions are dense in \(L^2(\mu)\).

If \(\mu\) is a Lebesgue-Stieltjes measure on \(\mathbb{R}\),

- simple functions of the form
  \[ S = \sum_{j=1}^{n} a_j \chi_{F_j} \quad F_j \supseteq \bigcup_{i=1}^{m} I_{ij} \]
  for \(\{I_{ij}\}_{i=1}^{m}\), disjoint open intervals are dense in \(L^2(\mu)\).

- \(C_c(\mathbb{R})\) is dense in \(L^2(\mu)\)
  \(\text{supp}(f) \subseteq \{x \in \mathbb{R} : f(x) \neq 0\}\) is compact.
Our proof uses the following result from HW 4:

Lemma: If \( \mu \) is a Lebesgue-Stieltjes measure and \( E \subset \text{cl} \mu^* \) with \( \mu(E) < +\infty \), then for all \( \varepsilon > 0 \), \( \exists \) disjoint open intervals \( E_i \) such that

\[
\mu(E \Delta \bigcup_{i=1}^{n} E_i) < \varepsilon.
\]

\[ A \triangle B = (A \setminus B) \cup (B \setminus A) \]

Proof of Thm: Fix \( f \in L^1(\mu) \).

Since \( f^+ \) and \( f^- \) are nonneg measurable functions, there exist simple fn's \( \psi_n \uparrow f^+ \) and \( s_n \uparrow f^- \).
Thus \( \Psi_n - S_n \to f^+ - f^- = f \) pointwise.

Furthermore,

\[
|\Psi_n - S_n - f| \leq \Psi_n + S_n + |f| \leq 2|f|
\]

Since \( f \in L^1 \), \( g = 2|f| \in L^1(\mu) \) is a dominating \underline{function}. Thus, by DCT

\[
\lim_{n \to \infty} \|\Psi_n - S_n - f\|_{L^1(\mu)} = \|\lim_{n \to \infty} (\Psi_n - S_n - f)\|_{L^1(\mu)} = 0
\]

This shows simple functions are dense in \( L^1(\mu) \).

Now, suppose \( \mu \) is a L-S measure on \( \mathbb{R} \).
Fix $\varepsilon > 0$. By what we just showed, $\exists \emptyset$ simple such that $\|\emptyset - \text{fill}\|_{L^1(\mu)} < \frac{\varepsilon}{2}$ and $|\emptyset| \leq |f|$. We may express $\emptyset = \sum_{j=1}^{n} a_j 1_{E_j}$, with $a_j \neq 0 \ \forall \ j$ and $\bigcap_{j=1}^{n} E_j$ are disjoint.

By construction, $\forall j$,$$
|a_j| \mu(E_j) \leq |\emptyset| \text{d}\mu \leq |f| \text{d}\mu < +\infty.
$$Thus, $\mu(E_j) < +\infty$. By Lemma, there exist disjoint open intervals $\bigcup_{i=1}^{m} I_i$ so that
\[ \mu(E_j \Delta \bigcup_{i=1}^{m_{ij}^*} I_{ij}^*) < \frac{\varepsilon}{2n \max_{j} |a_{ij}|} \]

Thus,
\[ \| \varnothing - \sum_{j=1}^{n} a_{ij} 1_{E_j \setminus \bigcup_{i=1}^{m_{ij}^*} I_{ij}^*} \|_{L^2(\mu)} \]
\[ = \sum_{j=1}^{n} |a_{ij}| \| 1_{E_j} - 1_{\bigcup_{i=1}^{m_{ij}^*} I_{ij}^*} \|_{L^2(\mu)} \]
\[ = \sum_{j=1}^{n} |a_{ij}| \mu(E_j \Delta \bigcup_{i=1}^{m_{ij}^*} I_{ij}^*) \]
\[ < \frac{\varepsilon}{2} \]

This shows "even simpler" fits are dense in \( L^2(\mu) \). Finish next time...