Recall: First Goal: define a function $\mu: 2^\mathbb{R} \to [0, +\infty]$ satisfying the following

1. If $\{E_i\}_{i=1}^n \subseteq 2^\mathbb{R}$ (or $\{E_i\}_{i=1}^\infty \subseteq 2^\mathbb{R}$) are disjoint, then
   \[
   \mu(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i) \quad \text{(or} \quad \mu(\bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty \mu(E_i))
   \]

Def: If $\mu: 2^\mathbb{R} \to [0, +\infty]$ satisfied 1, it is finitely (or countably) additive.

2. $\mu([a,b]) = b - a$

3. $\mu(E + c) = \mu(E) \quad \forall \ c \in \mathbb{R}, \ E \subseteq \mathbb{R}$
   \[
   = \{x+c : x \in E\}
   \]
Def: If \( \mu : 2^\mathbb{R} \to [0, +\infty] \) satisfies (3), it is translation invariant.

Thm: (Vitali) There is no function \( \mu : 2^\mathbb{R} \to [0, +\infty] \) satisfying (1), (2), and (3).

Lemma (monotonicity): Given a set \( X \) and \( \mu : 2^X \to [0, +\infty] \) finitely additive, then for all \( A, B \subseteq X \),
\[
A \subseteq B \implies \mu(A) \leq \mu(B).
\]
Which criterion do we weaken to get existence of such a measure?

If we weaken ① to finite additivity, there are still problems for δ≤3:

Banach-Tzarski Paradox (1924)

\[ U = \bigcup_{i=1}^{n} E_i \quad \text{O} \quad V = \bigcup_{i=1}^{n} F_i \]

Fi is a rotation/translation of Ei

Bogachev 1.12(xi)

What if we weaken criteria ② or ③? No longer compatible w/ usual notion of length.
Two good choices:

- don't require \( \mu \) to be defined on all of \( 2^{\mathbb{R}} \)
- still define \( \mu \) on all of \( 2^{\mathbb{R}} \), but replace countable additivity with countable sub-additivity:

\[
\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)
\]

We'll see that...

- given a measure on a "nice" family of subsets of \( \mathbb{R} \), it extends to an outer measure on \( 2^{\mathbb{R}} \)
- given an outer measure, we can single out "nice sets" on which it is a measure
What kind of family of subsets should we restrict to?

Let $X$ be a set.

**Definition:** $\mathcal{A} \subseteq 2^X$ is an algebra of subsets of $X$ if it is nonempty and

1. $E_1, \ldots, E_n \in \mathcal{A} \Rightarrow \bigcup_{i=1}^n E_i \in \mathcal{A}$
2. $E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A}$ \textit{“closed under complements”}

**Lemma:** If $\mathcal{A}$ is an algebra, then

1. $\emptyset \in \mathcal{A}, X \in \mathcal{A}$
2. $E_1, \ldots, E_n \in \mathcal{A} \Rightarrow \bigcap_{i=1}^n E_i \in \mathcal{A}$ \textit{“closed under finite intersections”}

**Proof:** HW2
Ex:
(i) \(\mathcal{A} = 2^X\)
(ii) \(\mathcal{A} = \{\emptyset, X\}\)
(iii) \(\mathcal{A} = \text{all finite subsets and all cofinite subsets}
\text{complements of finite sets}\)

Def: \(\mathcal{A} = 2^X\) is a \(\sigma\)-algebra of subsets of \(X\) if it is an algebra
and it is closed under countable unions.

Remark: A \(\sigma\)-algebra is also closed under countable intersections.

Remark: Any algebra that is closed under countable disjoint unions
is a \(\sigma\)-algebra. (HW 2)
Ex: (1) and (ii) are $\sigma$-algebras. (iii) isn't.

**Def:** Given a nonempty set $X$ and a $\sigma$-algebra $\mathcal{M} = 2^X$, we call $(X, \mathcal{M})$ a measurable space. We will call $E \subseteq \mathcal{M}$ a measurable set.

**Prop:** Given any $E \subseteq 2^X$, there exists a smallest $\sigma$-algebra $\mathcal{M}(E)$ containing $E$, known as the $\sigma$-algebra generated by $E$.

That is, all other $\sigma$-algebras $\mathcal{G}$ that contain $E$ satisfy $\mathcal{M}(E) \subseteq \mathcal{G}$.
Given any nonempty collection $C$ of $\sigma$-algebras on $X$, then

$$NC = \{E \subseteq X : E \in C \land \forall A \in C, E \cap A \in C \}$$

is a $\sigma$-algebra.

**Proof:**

Let $C = \{E : E$ is a $\sigma$-algebra on $X, E \subseteq \mathcal{X} \}$. Since $2^X \in C$, $C$ is nonempty.

By CLAIM, $NC$ is a $\sigma$-algebra.

By defn of $C$, $E \subseteq NC$, and for any $\sigma$-algebra $\mathcal{A}$ s.t. $E \subseteq \mathcal{A}$, $NC \subseteq \mathcal{A}$.

Thus, $\mathcal{M}(E) = NC$ is $\sigma$-algebra. q.e.d.
Rmk. Intuitively, $M(E)$ creates a $\sigma$-algebra containing all sets in $E$ by "going from the outside in," that is, starting with $\sigma$-algebras that are "too big" and taking intersections.

Recall: a topology $T$ is a collection of subsets of $X$ closed under arbitrary unions and finite intersections.

Let $(X, T)$ be a topological space.

Def. The Borel $\sigma$-algebra of $X$, denoted $B(X)$, is the $\sigma$-algebra generated by $T$. Its members are known as Borel sets.

What do the Borel sets look like? Let's go from the "inside out."
\( F_i = 2^X \)

\( \mathcal{F}^C = \{ \text{all countable unions of sets in } F_i \} \)

\( \mathcal{F}^S = \{ \text{all countable intersections of sets in } F_i \} \)

\( \overline{\mathcal{F}} = \{ \text{all complements of sets in } F \} \)

To build \( \mathcal{B}(X) \) from inside out:

\[ T \rightarrow T^S \rightarrow T^{S \cup T^S} \rightarrow \ldots \rightarrow \mathcal{B}(X) \]

"Borel hierarchy" uncountably many steps
Prop: The Borel σ-algebra of \( \mathbb{R} \), denote \( \mathcal{B}(\mathbb{R}) \), is generated by each of the following:

(i) open intervals \( E_1 = \{(a, b) : a < b\} \)
(ii) closed intervals \( E_2 = \{[a, b] : a < b\} \)
(iii) half-open intervals \( E_3 = \{(a, b] : a < b\} \)
(iv) open rays \( E_4 = \{(a, +\infty) : a \in \mathbb{R}\} \)
(v) closed rays \( E_5 = \{[a, +\infty) : a \in \mathbb{R}\} \)

Pf: HW 2
**Def:** A **measure** on a measurable space \((X, \mathcal{M})\) is a function
\(\mu : \mathcal{M} \rightarrow [0, +\infty]\), s.t.
(i) \(\mu(\emptyset) = 0\)
(ii) given \(\{E_i \}_{i=1}^\infty \subseteq \mathcal{M}\) disjoint
\[\mu\left(\bigcup_{i=1}^\infty E_i\right) = \sum_{i=1}^\infty \mu(E_i)\]

We call \((X, \mathcal{M}, \mu)\) a **measure space**.

**Ex:** (Dirac mass / Dirac measure)
\((X, \mathcal{M}) = (X, 2^X)\)
Fix \(x_0 \in X\) and define \(\mu(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{otherwise} \end{cases}\)

Often denoted \(\mu = \delta_{x_0}\).
\( \text{Ex(counting measure) } (X, \mathcal{M}) = (X, 2^X) \)
\[ \mu(A) = \# \text{ elements in } A \]

Thm: For any measure space \( (X, \mathcal{M}, \mu) \)
and \( A, B \in \mathcal{M} \), \( \exists A_i, i=1, \infty \in \mathcal{M} \),

(i) \( A \subseteq B \Rightarrow \mu(A) \leq \mu(B) \)

(ii) \( A \subseteq B, \mu(A) < +\infty \Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A) \) (not necessarily disjoint)

(iii) \( \mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i) \) (countable subadditivity)

(iv) \( A_i \subseteq A_{i+1}, \forall i \Rightarrow \mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i) \) (continuity from below)
\[(v) \quad A_{i+1} \subseteq A_i, \quad \forall i, \quad \mu(A_i) < +\infty \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu(A_i)\]

"continuity from above"

Remark: The hypothesis $\mu(A_i) < +\infty$ in $(v)$ is necessary.

Consider the counting measure on $(\mathbb{N}, 2^\mathbb{N})$.

Let $A_i = \{ n \in \mathbb{N} : n \leq i \}$, $\mu(A_1) = +\infty$

$0 = \mu(\emptyset) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right) \neq \lim_{i \to \infty} \mu(A_i) = +\infty$. 