Lecture 3
Recall: Measures

Let \( X \) be a set.

**Def:** \( \mathcal{A} = 2^X \) is an algebra of subsets of \( X \) if it is nonempty and

1. \( E_1, \ldots, E_n \in \mathcal{A} \Rightarrow \bigcup_{i=1}^n E_i \in \mathcal{A} \)
2. \( E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A} \)

**Lemma:** If \( \mathcal{A} \) is an algebra, then

1. \( \emptyset \in \mathcal{A}, X \in \mathcal{A}, \bigcap_{i=1}^n E_i \in \mathcal{A} \)
2. \( E_1, \ldots, E_n \in \mathcal{A} \Rightarrow \bigcap_{i=1}^n E_i \in \mathcal{A} \)
Def: \( \mathcal{A} = 2^X \) is a \( \sigma \)-algebra of subsets of \( X \) if it is an algebra and it is closed under countable unions.

Def: Given a nonempty set \( X \) and a \( \sigma \)-algebra \( \mathcal{M} = 2^X \), we call \((X, \mathcal{M})\) a measurable space. We will call \( E \in \mathcal{M} \) a measurable set.

Prop: Given any \( E \in 2^X \), there exists a smallest \( \sigma \)-algebra \( \mathcal{M}(E) \) containing \( E \), known as the \( \sigma \)-algebra generated by \( E \).

Let \((X, \mathcal{T})\) be a topological space.
Def: The Borel $\sigma$-algebra of $X$, denoted $\mathcal{B}(X)$ is the $\sigma$-algebra generated by $T$. Its members are known as Borel sets.

Prop: The Borel $\sigma$-algebra of $\mathbb{R}$, denote $\mathcal{B}(\mathbb{R})$, is generated by each of the following:

(i) open intervals $E_1 = \{(a, b): a < b\}$
(ii) closed " $E_2 = \{[a, b]: a < b\}$
(iii) half-open " $E_3 = \{(a, b]: a < b\}$
(iv) open rays $E_4 = \{(a, +\infty): a \in \mathbb{R}\}$
(v) closed " $E_5 = \{[a, +\infty): a \in \mathbb{R}\}$
Measures

Def: A measure on a measurable space \((X, \mathcal{M})\) is a function 
\(\mu : \mathcal{M} \rightarrow [0, +\infty] \) s.t.

(i) \(\mu(\emptyset) = 0\)

(ii) given \(\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}\) disjoint

\[\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)\]

We call \((X, \mathcal{M}, \mu)\) a measure space.
Theorem: For any measure space \((X, \mathcal{M}, \mu)\) and \(A, B \in \mathcal{M}\), \(\exists A_i, i \in \mathbb{N} \subseteq \mathcal{M},\)

(i) \(A \subseteq B \Rightarrow \mu(A) \leq \mu(B)\)

(ii) \(A \subseteq B, \mu(A) < +\infty \Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A)\) (not necessarily disjoint)

(iii) \(\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)\) "countable subadditivity"

(iv) \(A_i \subseteq A_{i+1}, \forall i \Rightarrow \mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i)\) "continuity from below"

(v) \(A_i \subseteq A_{i+1}, \forall i, \mu(A_i) < +\infty \Rightarrow \mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i)\) "continuity from above"
(Pg. 10) See monotonicity lemma in lec 1.

(ii) \( \mu(B) = \mu(A \cup (B\setminus A)) = \mu(A) + \mu(B\setminus A) \). The result follows if \( \mu(A) < \infty \).

(iii) Define \( B_1 = A_1 \), \( B_2 = A_2 \setminus A_1 \), \( B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i \). Then \( \bigcup_{i=1}^{\infty} A_i = \bigcup_{n=1}^{\infty} B_n \).

\[ \mu(\bigcup_{i=1}^{\infty} A_i) = \mu(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i). \]

(iv) Define \( B_1 = A_1 \), \( B_i = A_i \setminus A_{i-1} \). Then \( A_n = \bigcup_{i=1}^{n} B_i \Rightarrow \bigcup_{n=1}^{\infty} A_n = \bigcup_{i=1}^{\infty} B_i \).
Thus \( \mu(A_n) = \mu\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} \mu(B_i) \)

Consequently,

\[
\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{i \to \infty} \mu(A_i).
\]

(v): Define \( B_i = A_1 \setminus A_i \)

\[
B_1 \subseteq B_2 \subseteq B_3 \subseteq \ldots
\]

\[
\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} (A_1 \setminus A_i)
\]

\[
= \bigcup_{i=1}^{\infty} (A_1 \cap A_i^c)
\]

\[
= A_1 \cap \left(\bigcup_{i=1}^{\infty} A_i^c\right)
\]

\[
= A_1 \setminus (\bigcap_{i=1}^{\infty} A_i).
\]
\[ \mu(A_j) = \mu(A_j \setminus (\bigcap_{i=1}^{\infty} A_i)) \cup (\bigcap_{i=1}^{\infty} A_i) \]

\[ = \mu(A_j \setminus (\bigcap_{i=1}^{\infty} A_i)) + \mu(\bigcap_{i=1}^{\infty} A_i) \]

\[ = \mu(\bigcup_{i=1}^{\infty} B_i) + \mu(\bigcap_{i=1}^{\infty} A_i) \]

\[ = \lim_{i \to \infty} \mu(B_i) + \mu(\bigcap_{i=1}^{\infty} A_i) \]

\[ = \lim_{i \to \infty} \mu(A_j \setminus A_i) + \mu(\bigcap_{i=1}^{\infty} A_i) \]

\[ = \lim_{i \to \infty} \mu(A_j) - \mu(A_i) + \mu(\bigcap_{i=1}^{\infty} A_i) \]

Rearranging then gives the result.
Measure Terminology: \((\mathcal{X}, \mathcal{M}, \mu)\)

- \(\mu\) is a finite measure if \(\mu(\mathcal{X}) < +\infty\)

- \(\mu\) is a \(\sigma\)-finite measure if \(\exists \{E_i\}_i \subseteq \mathcal{M}\) s.t. \(\bigcup_i E_i = \mathcal{X}\) and \(\mu(E_i) < +\infty\).

\(\text{Ex: } (\mathcal{X}, \mathcal{M}) = (\mathbb{R}, \mathcal{B}(\mathbb{R})), E_i = B_i \cap 0)\)

- \(E\) is a null set (of \(\mu\)) if \(E \in \mathcal{M}\) and \(\mu(E) = 0\).

- We say that a property holds for \((\mu)\)-almost every \(x \in \mathcal{X}\) if the set of points where it doesn't hold is a null set.
Recall ultimate goal:
A measure $\mu$ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ where $\mu((a,b)) = b-a$ and it's translation invariant.

In order to prove such a measure exists...  
\[ \text{Outer measures} \]  
\[ \text{given that we all of } \mathbb{R} \]

**Def:** An outermeasure on a set $X$ is a function $\mu^* : 2^X \to [0, +\infty]$ s.t.

(i) $\mu^*(\emptyset) = 0$
(ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$
(iii) $\mu^*(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu^*(A_i)$

**Note:** not rec. disj
(Rmk: (ii) + (iii) $\implies$ If $E \subseteq \bigcup_{i=1}^{\infty} A_i$, then

\[ \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(A_i) \]

Ex: (Lebesgue outer measure)
Define $\mu: \mathcal{P}^{\infty} \rightarrow [0, \infty]$ by

\[ \mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} |b_i - a_i| : A \subseteq \bigcup_{i=1}^{\infty} [a_i, b_i] \right\} \]

- We will prove $\mu^*$ is an outermeasure.

- $\mu^*([a,b]) = b - a$

- $\mu^*$ is translation invariant

- Is it countably additive? No.
We will be able to show that it becomes countably additive when restricted to “nice enough”.

Which sets are “nice enough”?

\[ X \text{ nonempty set} \]
\[ \mu^* \text{ outer measure} \]

**Def:** \( A \subseteq X \) is \( \mu^* \)-measurable if

\[ (*) \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \forall E \subseteq X \]

Let \( \mathcal{M}_{\mu^*} = \{ A \subseteq X : A \text{ is } \mu^* \text{-measurable} \} \)

“A breaks apart any set \( E \) nicely”
Rmk: Suppose \( A \subseteq E \). Then (\( * \)) becomes

\[
\mu^*(E) = \mu^*(A) + \mu^*(E \setminus A)
\]

as long as everything finite...

\[
\mu^*(E) - \mu^*(E \setminus A) = \mu^*(A)
\]

outer approx

For inner measure: Bogachev i. 12(viii).

Rmk: Note that, for any \( A \subseteq X \), "\( \leq \)" always holds in (\( * \)) by subadditivity, so to check if \( A \in \mathcal{M}^* \), it suffices to check "\( \geq \)".
**Prop:** If \( \mu^*(B) = 0 \), then \( B \subseteq \mathcal{M} \).

**Pf:** For any \( E \subseteq X \),

\[
\mu^*(E) \geq \mu^*(E \cap B^c) + \mu^*(E \cap B^c) \tag{monotonicity}
\]

\[
\geq \mu^*(E \cap B) + \mu^*(E \cap B^c) \tag{monotonicity}
\]

**Thm (Carathéodory):** Given an outer measure \( \mu^* \) on \( X \),

(i) \( \mathcal{M} \) is a \( \sigma \)-algebra

(ii) \( \mu^* \) is a measure on \( \mathcal{M} \).

Q: Is this the “largest” \( \sigma \)-algebra on which \( \mu^* \) can be defined as a measure?

A: No. See HW3.
Prop: \( \mathcal{M}_\mu^* \) is an algebra and \( \mu^* \) is finitely additive on \( \mathcal{M}_\mu^* \).

**Proof:** \( \mathcal{M}_\mu^* \) is nonempty.

Since \( \mu^*(\emptyset) = 0 \Rightarrow \emptyset \in \mathcal{M}_\mu^* \)

\( \mathcal{M}_\mu^* \) is closed under complements.

By defn of \( \mu^* \)-meas set

\( \mathcal{M}_\mu^* \) is closed under finite unions.

It suffices to show \( A, B \in \mathcal{M}_\mu^* \Rightarrow A \cup B \in \mathcal{M}_\mu^* \).

Fix arbitrary \( E \subseteq X \).
\[ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \]
\[ = \mu^*(E \cap A) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \]
\[ \text{countably subadditive} \]
\[ \geq \mu^*((E \cap A) \cup (E \cap A^c \cap B)) + \mu^*(E \cap (A \cup B)^c) \]
\[ = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) \]

So \( A \cup B \in \mathcal{M} \mu^* \).

Thus \( \mathcal{M} \mu^* \) is an algebra.

Next: we will show something stronger than \( \mu^* \left|_{\mathcal{M}} \mu^* \) is finitely additive. \]
Claim 1: Given $\bigcup_{i=1}^{n} B_i \subseteq \mathcal{M}$ disjoint, for all $E \subseteq X$,

$$\mu^*(E \cap (\bigcup_{i=1}^{n} B_i)) = \sum_{i=1}^{n} \mu^*(E \cap B_i)$$

By next time...