Recall:

**Measure Terminology** $(X, \mathcal{M}, \mu)$

- $\mu$ is a **finite measure** if $\mu(X) < +\infty$.

- $\mu$ is a **$\sigma$-finite measure** if $\exists \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ s.t. $\bigcup_{i=1}^{\infty} E_i = X$ and $\mu(E_i) < +\infty$.

- $E$ is a null set (of $\mu$) if $E \in \mathcal{M}$ and $\mu(E) = 0$.

- We say that a property holds for $(\mu)$-almost every $x \in X$ if the set of points where it doesn’t hold is a null set.

- $\mu$ is a **Borel measure** if it is a measure on $\mathcal{B}_X$, for top space $(X, T)$.
**Outer measures**

**Def:** An outer measure on a set $X$ is a function $\mu^*: 2^X \rightarrow [0, +\infty]$ s.t.
1. $\mu^*(\emptyset) = 0$
2. $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$
3. $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$

not rec. disj

**Ex:** (Lebesgue outer measure)
Define $\mu: 2^\mathbb{R} \rightarrow [0, +\infty]$ by

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} |b_i - a_i| : A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}$$

- $\mu^*([a, b]) = b - a$
- $\mu^*$ is translation invariant
- We will prove $\mu^*$ is an outer measure.
We will be able to show that it becomes countably additive when restricted to “nice enough”.

Which sets are “nice enough”? (Hopefully Borel sets!)

Let $X$ be a nonempty set and $\mu^*$ an outer measure.

**Def**: $A \subseteq X$ is $\mu^*$-measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \forall E \subseteq X$$

Let $\mathcal{M}_{\mu^*} = \{A \subseteq X : A \text{ is } \mu^*-\text{measurable}\}$.

“A breaks apart any set $E$ nicely.”
Prop: If $\mu^*(B) = 0$, then $B \in \mathcal{M}^\mu$.

Thm (Carathéodory): Given an outer measure $\mu^*$ on $X$,

(i) $\mathcal{M}^\mu$ is a $\sigma$-algebra

(ii) $\mu^*$ is a measure on $\mathcal{M}^\mu$

Prop: $\mathcal{M}^\mu$ is an algebra and $\mu^*$ is finitely additive on $\mathcal{M}^\mu$.

Pf of Prop:

Last time: $\mathcal{M}^\mu$ is an algebra.

Next: we will show something stronger than $\mu^*/\mathcal{M}^\mu$ is finitely additive.
**Claim 1:** Given \( \{B_i\}_{i=1}^n \subseteq \mathcal{M} \mu^* \) disjoint, for all \( E \subseteq X \),

\[
\mu^*(E \cap (\bigcup_{i=1}^n B_i)) = \sum_{i=1}^n \mu^*(E \cap B_i).
\]

Then, taking \( E = X \) gives \( \mu^* \) finitely additive.

**Proof of Claim 1:**

Base case: \( n = 1 \)

Inductive step: Assume equality holds for \( n-1 \).

\[
\mu^*(E \cap (\bigcup_{i=1}^n B_i)) \downarrow \text{since } B_n \in \mathcal{M} \mu^*
\]

\[
= \mu^*(E \cap (\bigcup_{i=1}^{n-1} B_i) \cap B_n) + \mu^*(E \cap (\bigcup_{i=1}^{n-1} B_i) \cap (\bigcup_{i=1}^n B_i) \cap B_n)
\]

+ \mu^*(E \cap (\bigcup_{i=1}^n B_i) \cap B_n^c)
We will actually prove something stronger...

Claim 2: Given $\exists B_{i=1}^{\infty} \subseteq \mathcal{M} \mu^*$ disjoint for all $E \subseteq \mathcal{X}$

$$
\mu^*(E) = \sum_{i=1}^{\infty} \mu^*(ENB_i) + \mu^*(E \cap (\bigcup_{i=1}^{\infty} B_i))
$$
In particular, for \( E = \bigcup_{i=1}^{\infty} B_i \), this implies countable additivity.

Proof of Claim 2:

"\( \leq \)" follows by countable subadditivity, since
\[
E = (\bigcup_{i=1}^{\infty} (\bigcap_{j=i}^{\infty} B_j)) \cup (\bigcap_{i=1}^{\infty} (\bigcup_{j=i}^{\infty} B_j)^c)
\]

It remains to show "\( \geq \)".

Since \( \mu^* \) is closed under finite unions, \( \bigcup_{i=1}^{n} B_i \in \Sigma_{\mu^*} \), so by defn of \( \mu^* \)-meas.
\[ \mu^*(E) = \mu^*(E \cap \bigcup_{i=1}^{\infty} B_i) + \mu^*(E \cap (\bigcup_{i=1}^{\infty} B_i)^c) \]

Taking \( n \to +\infty \) gives Claim \( \theta \).

\( \mu^* \) is closed under countable unions.

It suffices to show it is closed under countable disjoint unions. Given \( \{B_i\}_{i=1}^{\infty} \subseteq \mathcal{M} \) disjoint. By Claim \( \theta \), \( \forall E \subseteq X \)

\[ \mu^*(E) = \sum_{i=1}^{\infty} \mu^*(E \cap B_i) + \mu^*(E \cap (\bigcup_{i=1}^{\infty} B_i)^c) \geq \mu^*(\bigcup_{i=1}^{\infty} (E \cap B_i)) + \mu^*(E \cap (\bigcup_{i=1}^{\infty} B_i)^c) = \mu^*(E \cap (\bigcup_{i=1}^{\infty} B_i)) + \mu^*(E \cap (\bigcup_{i=1}^{\infty} B_i)^c) \]
Thus $i = 1, B_i \in \mathcal{M}_{\mu^*}$.

Back to Lebesgue outer measure

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} |b_i - a_i| : A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}$$

We will study a generalization of this that gives rise to Lebesgue-Stieltjes measures.

Recall: $F: \mathbb{R} \rightarrow [0, \infty]$ is right-continuous if, for all $x \in \mathbb{R}$,

$$\lim_{y \to x^+} F(y) = F(x)$$

\[ \lim_{\uparrow} \] \[ \lim_{\uparrow} \]
**Def:** Given $F: \mathbb{R} \to \mathbb{R}$ non-decreasing and right continuous, define $\mu^*_F: \mathcal{B} \to [0, +\infty]$ \[ \mu^*_F(A) = \inf \left\{ \sum_{i=1}^{\infty} I_i : A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \right\} \]

Why do we require $F$ non-decreasing and right cts?

**Spoiler:** We will show that any finite measure $\mu$ on $\mathcal{B}$ satisfies $\mu = \mu^*_F |_{\mathcal{B}}$ for

$$F(x) = \mu((\infty, x])$$

$F$ is the cumulative distribution function of $\mu$. 
Note that if $\mu$ is a finite measure on $\mathbb{R}$ and $F(x)$ is its CDF,

- $F(x)$ is nondecreasing: 
  \[ x \leq y \Rightarrow (-\infty, x] \subseteq (-\infty, y] \Rightarrow F(x) \leq F(y) \]

- $F(x)$ is right continuous: 
  For any sequence $x_n \downarrow x$, 
  \[
  \lim_{n \to \infty} F(x_n) = \lim_{n \to \infty} \mu((-\infty, x_n])
  \]

From above, 
\[
\mu \text{ finite measure}
\]
\[
= \mu((-\infty, x])
\]
\[= F(x) \]
Thm: $\mu^*_F$ is an outer measure.

Proof:
$\mu^*_F(\emptyset) = 0$ since $\emptyset \subseteq (0,0)$, by defn of $\mu^*_F$,
\[ 0 \leq \mu^*_F(\emptyset) \leq F(0) - F(0) = 0. \]

Now we show $A \subseteq \bigcup_{j=1}^{\infty} B_j \Rightarrow \mu^*_F(A) \leq \sum_{j=1}^{\infty} \mu^*_F(B_j)$

WLOG $\mu^*_F(B_j) < \infty \quad \forall j$. 
By defn of \( \inf \), \( \forall \varepsilon > 0, j = 1, \ldots \)
\[ \exists \{ \mathbb{I}^{j, i}_{i=1} \}_{i=1}^\infty s.t. \]
- \( B_j \subseteq \bigcup_{i=1}^\infty \mathbb{I}^{j, i} \)
- \( m^*_F(B_j) \leq \sum_{i=1}^\infty |\mathbb{I}^{j, i}|_F = m^*_F(B_j) + \frac{\varepsilon}{2^j} \)

Since \( A \subseteq \bigcup_{j=1}^\infty B_j \), \( A \subseteq \bigcup_{i=1}^\infty \bigcup_{j=1}^\infty \mathbb{I}^{j, i} \). So
\[ m^*_F(A) \leq \sum_{i=1}^\infty |\mathbb{I}^{j, i}|_F \leq \sum_{i=1}^\infty m^*_F(B_j) + \frac{\varepsilon}{2^j} \]
\[ = \sum_{j=1}^\infty m^*_F(B_j) + \varepsilon \]

Sending \( \varepsilon \to 0 \) gives the result. \( \square \)
Thm: For all \( a, b \in \mathbb{R}, a \leq b, \)
\[ m^* F((a, b]) = F(b) - F(a). \]

Of: "\( \leq \)" follows quickly, since \( (a, b] \subseteq (a, b] \cup (\mathbb{Z} \cup ...), \) so by defn \( m^* F((a, b]) \leq F(b) - F(a) + 0 + 0 + ... \)
Reverse inequality next time : (