Math 201a: Midterm 2 Practice Problems
(Not to be turned in)

For sake of time, on an actual midterm, I would have given you Q1a and Q3a as facts, rather than asking you to prove them.

Question 1

Consider a measure space \((X, \mathcal{M}, \mu)\) with \(\mu(X) < +\infty\).

In this question, you will prove that convergence in measure is metrizable: there exists a metric on the space of measurable functions (up to almost everywhere equivalence) so that convergence in this metric is equivalent to convergence in measure.

(a) Define \(\phi : [0, +\infty) \to [0, +\infty)\) by \(\phi(s) = s/(1 + s)\). Prove that \(\phi(s)\) is nondecreasing, \(\phi(s + t) \leq \phi(s) + \phi(t)\) and \(\phi(s) = 0 \iff s = 0\).

(b) Given \(f, g : X \to \mathbb{R}\) measurable, define

\[\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|} \, d\mu.\]

Prove that \(\rho\) is a metric on the space of measurable functions, if we identify functions that are equal almost everywhere.

(c) Given \(f_n, f : X \to \mathbb{R}\) measurable, show that

\(\rho(f_n, f) \to 0 \iff f_n \to f\) in measure.

Question 2

Suppose \((X, \mathcal{M}, \mu)\) is a measure space and \(f_n, f\) are nonnegative \((\mathcal{M}, \mathcal{B}_\mathbb{R})\)-measurable functions satisfying \(f_n(x) \to f(x)\) for all \(x \in X\). Consider \(S \in \mathcal{M}\) arbitrary.

(a) Prove that

\[\int_{\mathbb{R}} 1_S f \, d\mu \leq \liminf_{n \to +\infty} \int_{\mathbb{R}} 1_S f_n \, d\mu \quad \text{and} \quad \int_{\mathbb{R}} 1_S f \, d\mu \leq \liminf_{n \to +\infty} \int_{\mathbb{R}} 1_S f_n \, d\mu.\]

(b) Use part (a) to show that,

\[\lim_{n \to +\infty} \int_{\mathbb{R}} f_n \, d\mu = \int_{\mathbb{R}} f \, d\mu < +\infty \iff \lim_{n \to +\infty} \int_{S} f_n \, d\mu = \int_{S} f \, d\mu, \forall S \in \mathcal{M}.\]

Question 3

(a) Suppose \((X, d)\) is an arbitrary metric space. Assume \(\{x_n\}_{n=1}^{+\infty} \subseteq X\) satisfies

\[\exists x \in X \text{ s.t. every subsequence } x_{n_k} \text{ has a further subsequence } x_{n_{kl}} \text{ for which } \lim_{l \to +\infty} d(x_{n_{kl}}, x) = 0.\]

Prove that \(\lim_{n \to +\infty} d(x_n, x) = 0\).

(b) Suppose \(f_n \to f\) in measure and there exists \(g \in L^1(\mu)\) so that \(|f_n| \leq g\) \(\mu\)-a.e. for all \(n \in \mathbb{N}\). Prove that \(f_n \to f\) in \(L^1(\mu)\).