Math 201a: Practice Midterm 1
(not to be turned in)

Question 1
Fix a measurable space \((X, \mathcal{M})\). Given \(f, g : X \rightarrow \mathbb{R}\) measurable, prove the following:

(a) \(f + g\) is measurable
(b) \(f^2\) is measurable
(c) \(fg\) is measurable (Hint: how can you express this in terms of squares and use part (b)?)

Question 2
Suppose \(\mu\) and \(\nu\) are finite measures defined on the same set and \(\sigma\)-algebra \((X, \mathcal{M})\). Prove that there exists a set \(N \in \mathcal{M}\) with the following properties:

(i) \(\mu(N) = 0\);
(ii) if \(S \in \mathcal{M}, S \subseteq X \setminus N\), and \(\mu(S) = 0\), then \(\nu(S) = 0\).

Hint: among all sets \(N \in \mathcal{M}\) with \(\mu(N) = 0\), choose the one for which \(\nu(N)\) is largest.

Question 3
In HW3, Q3, you showed that, given an outer measure \(\mu^*\), the collection of \(\mu^*\)-measurable sets \(\mathcal{M}_{\mu^*}\) is not necessarily the largest \(\sigma\)-algebra on which \(\mu^*\) can be restricted to be a measure. In this problem, you will show that, as long as the outer measure of any subset can be approximated by a \(\mu^*\)-measurable set containing it, then the collection of \(\mu^*\) measurable sets is maximal.

Let \(X\) be a nonempty set and suppose \(\mu^*\) is an outer measure on \(X\). Suppose that, for all \(S \subseteq X\) and for all \(\epsilon > 0\), there exists a \(\mu^*\)-measurable set \(E \supseteq S\) so that \(\mu^*(E) \leq \mu^*(S) + \epsilon\).

(a) Suppose \(A\) is not \(\mu^*\)-measurable and consider the \(\sigma\)-algebra \(\mathcal{F}\) generated by \(\mathcal{M}_{\mu^*}\) and \(\{A\}\). Prove that \(\mu^*\) is not additive on \(\mathcal{F}\).

(b) Use part (a) to conclude that \(\mathcal{M}_{\mu^*}\) is the largest \(\sigma\)-algebra on which \(\mu^*\) can be restricted to be a measure. (Hint: this is almost immediate from part (a).)