MATH 201A: HOMEWORK 1

Due Friday, October 4th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice questions will appear on each exam. You are encouraged to use the results of previous homework problems to solve subsequent homework problems.

We begin by recalling the following facts:

(i) Recall that, on a topological space X, a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converges to a limit $x \in X$ if, for any open set U containing x, there exists $N \in \mathbb{N}$ so that $x_n \in U$ for all $n \geq N$.

(ii) Recall that, for any real valued sequence $\{x_n\}_{n\in\mathbb{N}}$

$$x_n \text{ converges } \iff \limsup_{n\to+\infty} x_n = \liminf_{n\to+\infty} x_n.$$

Furthermore, if either equivalent condition holds, then $x_* = \limsup_{n\to+\infty} x_n = \liminf_{n\to+\infty} x_n$ is the limit of x_n.

(iii) For any nonempy subset $S \subseteq \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$, we say $M \in \overline{\mathbb{R}}$ is an *upper bound* of S if $M \geq s$ for all $s \in S$, and we define $\sup(S)$ to the the *least upper bound* of S. If $S = \emptyset$, we define $\sup(S) = -\infty$. In an analogous way, we may define the infimum of S. It follows immediately from the definition that, if $-S := \{-s : S \in S\}$, then $\sup(-S) = -\inf(S)$ and $\inf(-S) = -\sup(S)$.

Question 1

Given a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathbb{R}}$, define

$$\limsup_{n\to+\infty} x_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} x_n \quad \text{and} \quad \liminf_{n\to+\infty} x_n = \sup_{k\in\mathbb{N}} \inf_{n\geq k} x_n.$$

(a) Prove that $\liminf_{n\to+\infty} x_n \leq \limsup_{n\to+\infty} x_n$.

(b) For any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathbb{R}}$, prove that

$$\limsup_{n\to+\infty}(-x_n) = -\liminf_{n\to+\infty} x_n.$$

(c) For any sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \in \overline{\mathbb{R}}$, prove that

$$\limsup_{n\to+\infty}(x_n + y_n) \leq \limsup_{n\to+\infty} x_n + \limsup_{n\to+\infty} y_n,$$

as long as none of the sums are of the form $\infty - \infty$. Give an example where strict inequality holds.

(d) If $x_n \leq y_n$ for all n, prove that

$$\liminf_{n\to+\infty} x_n \leq \liminf_{n\to+\infty} y_n.$$

Question 2*

First, we recall some basic facts about the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$. We endow $\overline{\mathbb{R}}$ with the topology generated by sets of the following form, for $a, b \in \mathbb{R}$,

$$(a,b) := \{x \in \overline{\mathbb{R}} : a < x < b\}, \ \ (a,+\infty] := \{x \in \overline{\mathbb{R}} : x > a\}, \ \ [-\infty, b) := \{x \in \overline{\mathbb{R}} : x < b\}, \ \ \text{for } a, b \in \overline{\mathbb{R}}.$$

In other words, any open set in $\overline{\mathbb{R}}$ can be expressed as a union of sets of the above form.

(a) It is clear from the definition that, for any real valued sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$, if x_n converges to some $x_* \in \mathbb{R}$ with respect to the usual topology on $\mathbb{R}$, then $\{x_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathbb{R}}$ converges with respect to the topology on $\overline{\mathbb{R}}$.

Prove that, if $\{x_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathbb{R}}$ converges to $x_* \in \mathbb{R}$, then, up to modifying finitely many elements of the sequence, we have $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and x_n converges to x_* in the usual topology on $\mathbb{R}$.

(b) Prove that a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathbb{R}}$ converges with respect to the topology on $\overline{\mathbb{R}}$ if and only if $\liminf_{n\to+\infty} x_n = \limsup_{n\to+\infty} x_n$. Furthermore, if either equivalent condition holds, then $x_* = \liminf_{n\to+\infty} x_n = \limsup_{n\to+\infty} x_n = \lim_{n\to+\infty} x_n$ is the limit of x_n. (You may use, without proof, the analogous fact for real valued sequences, as recalled above.)

The terminology "converges with respect to the topology on $\overline{\mathbb{R}}$" is a bit counterintuitive. (For example, $x_n = n$ would "converge" to $+\infty$.) Consequently, in practice, we will say that $x_n \in \overline{\mathbb{R}}$ *has a limit* or *the limit exists* whenever it converges with respect to the topology on $\overline{\mathbb{R}}$, and we will write $\lim_{n\to+\infty} x_n$ for the limiting value.

(c) Prove that $\varphi : \overline{\mathbb{R}} \to [-1,1] : x \mapsto x/(1+|x|)$ is a homeomorphism, when $[-1,1]$ is endowed with the usual topology.

As a consequence of part (c), we conclude that $\overline{\mathbb{R}}$ is a compact metric space. Consequently, for any other metric space (X,d), a function $f : X \to \overline{\mathbb{R}}$ is continuous if and only if, for all convergent sequences x_n, $\lim_{n\to+\infty} f(x_n) = f(\lim_{n\to+\infty} x_n)$.

Question 3*

Given a metric space (X,d), we say that...

- $f : X \to \overline{\mathbb{R}}$ is *lower semicontinuous* in case $\{x : f(x) > a\}$ is open for all $a \in \overline{\mathbb{R}}$;
- $f : X \to \overline{\mathbb{R}}$ is *upper semicontinuous* in case $\{x : f(x) < a\}$ is open for all $a \in \overline{\mathbb{R}}$.

(a) Prove that $f : X \to \overline{\mathbb{R}}$ is lower semicontinuous if and only if, for all $x_0 \in X$ and every sequence x_n converging to x_0, we have

$$f(x_0) \leq \liminf_{n\to+\infty} f(x_n).$$

(b) Prove that $f : X \to \overline{\mathbb{R}}$ is upper semicontinuous if and only if, for all $x_0 \in X$ and every sequence x_n converging to x_0, we have

$$f(x_0) \geq \limsup_{n\to+\infty} f(x_n).$$

(c) Prove that $f : X \to \overline{\mathbb{R}}$ is continuous if and only if f is both upper and lower semicontinuous.

Question 4*

We begin by recalling the definition of the Riemann integral from undergraduate analysis. Fix an interval $[a,b]$, $a \neq b$. A *partition* P of $[a,b]$ is a finite set of points $x_0, x_1, \dots x_n$ satisfying

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Define $\Delta x_i = x_i - x_{i-1}$. For any bounded, real valued function $f : [a,b] \to \mathbb{R}$, we may define the *upper and lower sums* with respect to a given partition P:

$$U(P,f) := \sum_{i=1}^{n} M_i \Delta x_i \ , \quad M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x),$$
$$L(P,f) := \sum_{i=1}^{n} m_i \Delta x_i \ , \quad m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x).$$

Finally, the *upper and lower Riemann integrals* of f over $[a,b]$ are defined by

$$\overline{\int_a^b} f(x)dx = \inf_P U(P,f)$$
$$\underline{\int_a^b} f(x)dx = \sup_P L(P,f).$$

If $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$, then we say f is *Riemann integrable on* $[a,b]$, and the value of its integral is given by

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

Let $f : [0,1] \to \mathbb{R}$ be the function that is 1 for every rational number and 0 for every irrational number. Prove that f is not Riemann integrable on $[0,1]$.

Question 5

(a) Let $C([0,1])$ denote the space of continuous, real valued functions on the interval $[0,1]$. For all piecewise continuous functions $f, g \in C_{p.w.}([0,1])$, define

$$d_\infty(f,g) := \sup_{x \in [0,1]} |f(x) - g(x)|$$
$$d_1(f,g) := \int_0^1 |f(x) - g(x)| dx,$$

where the integral is the Riemann integral. Prove that $(C([0,1]), d_\infty)$ and $(C([0,1]), d_1)$ are metric spaces.

(b) Consider the sequence of functions

$$f_n(x) := \begin{cases} 1 & \text{if } x \in \left[0, \frac{1}{2}\right] \\ 1 - n(x - \frac{1}{2}) & \text{if } x \in \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{n}\right) \\ 0 & \text{if } x \in \left[\frac{1}{2} + \frac{1}{n}, 1\right]. \end{cases} \tag{1}$$

Is f_n a Cauchy sequence w.r.t. d_∞? Is f_n a Cauchy sequence w.r.t. d_1?

(c) In undergraduate analysis, you learned that $(C([0,1]), d_\infty)$ is complete. Is $(C([0,1]), d_1)$ complete?

Hint: Assume that f_n converges to some $f \in C([0,1])$ and prove that we must have $f(x) = 1$ for $x \geq 1/2$ and $f(x) = 0$ for $x \leq 1/2$.

Question 6*

(a) Define an equivalence relation on $\mathbb{R}$ as follows: $x \sim y \iff x - y \in \mathbb{Q}$. Prove that every equivalence class contains elements in the interval $[0,1]$.

(b) Using the equivalence relation from part (a) and the Axiom of Choice, define a set A by choosing one element in $[0,1]$ for each equivalence class. Prove that $\{A+q\}_{q \in \mathbb{Q} \cap [-1,1]}$ is a disjoint collection of sets.

Question 7*

Given a metric space (X,d) and a function $f : X \to \overline{\mathbb{R}}$, define its *lower semicontinuous envelope* to be

$$f_*(x) := \lim_{\epsilon \to 0} \left(\inf\{ f(y) : d(x,y) < \epsilon \} \right).$$

(a) Why does the limit $\epsilon \to 0$ exist for all $x \in X$?

(b) Prove that f_* is lower semicontinuous.

(c) Prove that $f_*(x) \leq f(x)$ for all $x \in X$.

(d) Prove that $f_*(x) = \inf\{\liminf_{n \to +\infty} f(x_n) : x_n \to x\}$ for all $x \in X$.

Question 8

Consider a metric space (X,d).

(a) Prove that the supremum of any collection of lower semicontinuous functions on (X,d) is lower semicontinuous.

(b) Prove that if $g(x)$ is a lower semicontinuous function satisfying $g(x) \leq f(x)$ for all $x \in X$, then $g(x) \leq f_*(x)$.

(c) Prove that

$$f_*(x) = \sup\{ g(x) : g : X \to \overline{\mathbb{R}} \text{ is lower semicontinuous and } g(x) \leq f(x)\ \forall x \in X\}.$$

Question 9*

Consider nonempty sets X, Y, and a function $f : X \to Y$.

(a) Prove that $f^{-1}\left(\cup_{\alpha \in A} E_\alpha\right) = \cup_{\alpha \in A} f^{-1}(E_\alpha),\ \forall E_\alpha \subseteq Y$.

(b) Prove that $f^{-1}(E^c) = \left(f^{-1}(E)\right)^c,\ \forall E \subseteq Y$.

(c) Which is always true? Which is not always true?

$$f\left(\cup_{\alpha \in A} E_\alpha\right) = \cup_{\alpha \in A} f(E_\alpha),\ \forall E_\alpha \subseteq X, \text{ or } f(E^c) = (f(E))^c,\ \forall E \subseteq X$$

Justify your answer.

