Homework 1 Solutions
Math 201a, F24

(2)

(a) Suppose $\{x_n\}_{n \in \mathbb{N}} \subseteq \overline{\mathbb{R}}$ converges to $x_* \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_*-\varepsilon, x_*+\varepsilon)$ is an open set containing x_*, $\exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq N_\varepsilon$, $x_n \in (x_*-\varepsilon, x_*+\varepsilon) \subseteq \mathbb{R}$. Thus, up to modifying the first $N-1$ elements of $\{x_n\}_{n \in \mathbb{N}}$, we have $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$. Furthermore, as above, $x_n \to x_*$ with respect to the usual topology on $\mathbb{R}$.

(b)

First suppose $\{x_n\}_{n \in \mathbb{N}} \subseteq \overline{\mathbb{R}}$ converges to $x_* \in \overline{\mathbb{R}}$.

$\boxed{\text{Case 1}}$: $x_* \in \mathbb{R}$

By part (a), up to modifying finitely many elements of x_n, we have $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ and $x_n \to x_*$ in $\mathbb{R}$. Let $\tilde{x}_n$ denote the modified sequence.

Thus, $\limsup_{n\to\infty} \tilde{x}_n = \liminf_{n\to\infty} \tilde{x}_n = x_*$.

Finally, since $k \mapsto \sup_{n\geq k} \tilde{x}_n$ is decreasing,

$$\limsup_{n\to\infty} \tilde{x}_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} \tilde{x}_n = \inf_{\substack{k\in\mathbb{N} \\ k\geq N}} \sup_{n\geq k} \tilde{x},$$

for any $N\in\mathbb{N}$. Thus $\limsup_{n\to\infty} \tilde{x}_n = \limsup_{n\to\infty} x_n$. Likewise, $\liminf_{n\to\infty} \tilde{x}_n = \liminf_{n\to\infty} x_n$. This gives the result.

$\boxed{\text{Case 2}}$: $x_* = +\infty$

By definition of convergence, for all $l\in\mathbb{N}$, $\exists\, N_l$ s.t. $n\geq N_l$ ensures $x_n \in (l, +\infty]$.

Thus,

$$l \leq \inf_{n\geq N_l} x_n \Rightarrow l \leq \sup_{k\in\mathbb{N}} \inf_{n\geq k} x_n$$

Hence, since l was arbitrary, $\liminf_{n\to\infty} x_n = +\infty$.

Finally, by ①ⓐ, $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$, which gives the result.

$\boxed{\text{Case 3}}$: $x_* = -\infty$

By definition of convergence, $-x_n \to +\infty$.
By the previous case, $\liminf_{n\to\infty} -x_n = \limsup_{n\to\infty} x_n = +\infty$.
By (1)(b), we obtain the result.

Now suppose $\{x_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathbb{R}}$ satisfies $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$, and define $x_* = \limsup_{n\to\infty} x_n$.
We will show x_n converges to x_*.

$\boxed{\text{Case 1}}$ $x_* \in \mathbb{R}$

By definition of infimum/supremum, since this cannot be a lower bound for $\{\sup_{n\geq k} x_n : k\in\mathbb{N}\}$
$\exists k_1, k_2 \in \mathbb{N}$ s.t.

$$x_* - 1 \leq \inf_{n\geq k_2} x_n, \quad \sup_{n\geq k_1} x_n \leq x_* + 1.$$

Thus, up to modifying finitely many elements of x_n, we see the sequence is real valued. Let $\tilde{x}_n$ denote the modified sequence. As argued above, modifying finitely many values of a sequence does not change its liminf or limsup, so
$\liminf_{n\to\infty} \tilde{x}_n = \limsup_{n\to\infty} \tilde{x}_n = x_*$. This implies $\tilde{x}_n$ converges to x_*. Hence, so does x_n.

$\boxed{\text{Case 2}}$ $x_* = +\infty$

Let $U \subseteq \overline{\mathbb{R}}$ be an arbitrary open set containing $+\infty$. By definition of the topology on $\overline{\mathbb{R}}$, there exists $a \in \mathbb{R}$ s.t. $(a, +\infty] \subseteq U$. Since $\liminf_{n\to\infty} x_n = +\infty$, a is not an upper bound for $\{\inf_{n\geq k} x_n : k \in \mathbb{N}\}$ and $\exists\ k_1 \in \mathbb{N}$ s.t. $a < \inf_{n\geq k_1} x_n$. Thus $\forall n \geq k_1$, $x_n \in (a, +\infty] \subseteq U$. This shows x_n converges to $x_* = +\infty$.

$\boxed{\text{Case 3}}$ The result follows from Case 2, by considering the sequence $-x_n$.

(b)

First, we show φ is continuous. It suffices to show that for any $x \in \overline{\mathbb{R}}$ and $0 < \varepsilon < 1$, there exists an open subset U of $\overline{\mathbb{R}}$ so that $y \in U$ ensures $|\varphi(x) - \varphi(y)| < \varepsilon$.

If $x=+\infty$ and $y \in \{z : z > (\frac{1}{1-\varepsilon}-1)^{-1}\}$, then $y>0$ and

$$\frac{1}{y} < \frac{1}{1-\varepsilon} - 1 \Rightarrow \frac{1}{y}(1+y) < \frac{1}{1-\varepsilon} \Rightarrow 1-\varepsilon < \frac{y}{1+y} = \frac{y}{1+|y|},$$

so $|\varphi(x)-\varphi(y)| < \varepsilon$. Similarly, if $x=-\infty$ and $y \in \{z : z < (\frac{1}{\varepsilon-1}+1)^{-1}\}$, then $y<0$ and

$$\frac{1}{y} > 1 + \frac{1}{\varepsilon-1} \Rightarrow \frac{1}{y}(1-y) > \frac{1}{-1+\varepsilon} \Rightarrow \frac{y}{1+|y|} = \frac{y}{1-y} < -1+\varepsilon,$$

so $|\varphi(x)-\varphi(y)| < \varepsilon$.

If $x \in \mathbb{R}$,

$$|\varphi(x)-\varphi(y)| = \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right| = \left|\frac{x+x|y|-y-y|x|}{1+|y|+|x|+|xy|}\right|$$
$$\le |(x-y) + x|y| - y|x||$$
$$\le |x-y| + |x|y| - y|x||.$$

If $x=0$, $|x-y|<\varepsilon$ ensures $|\varphi(x)-\varphi(y)|<\varepsilon$.
If $x \in \mathbb{R}\setminus\{0\}$, choose $\delta>0$ so that $\delta < \min\{\frac{|x|}{2}, \varepsilon\}$. Then $|x-y|<\delta$ ensures $\operatorname{sgn}(x)=\operatorname{sgn}(y)$, so $|\varphi(x)-\varphi(y)| \le |x-y| < \delta < \varepsilon$.

This shows φ is continuous.

Furthermore, we see that $x<y$ implies $x+x|y| < y+y|x|$, since either x and y have the same sign ($x|y| = y|x|$), or x is nonpositive and y is positive ($x|y| \le y|x|$). Thus $x<y \Rightarrow x(1+|y|) < y(1+|x|) \Leftrightarrow \varphi(x) < \varphi(y)$.

This shows φ is strictly increasing, hence injective.

Since $\varphi(-\infty) = -1$ and $\varphi(+\infty) = 1$, by the intermediate value theorem, φ is surjective onto $[-1,1]$.

φ^{-1} is also strictly increasing: if $s,t \in [-1,1]$ satisfy $s<t$, there exist $x,y \in \overline{\mathbb{R}}$ so that $\varphi(x) = s$ and $\varphi(y) = t$. Since $s<t$, we must have $\varphi^{-1}(s) = x < y = \varphi^{-1}(t)$.

Finally, we show φ^{-1} is continuous. It suffices to show that, for any $s \in [-1,1]$ and any set U of the form $(a,+\infty]$, $[-\infty,b)$, or (a,b), for $a,b \in \mathbb{R}$, containing $\varphi^{-1}(s)$, there exists $\varepsilon > 0$ so that $|t-s| < \varepsilon$, $t \in [-1,1]$ ensures $\varphi^{-1}(t) \in U$.

If $\varphi^{-1}(s) \in (a,+\infty]$, $\varphi^{-1}(s) > a \Rightarrow s > \varphi(a)$. Choose $\varepsilon_a > 0$ so that $|t-s| < \varepsilon_a$ ensures $t > \varphi(a)$ $\Rightarrow \varphi^{-1}(t) \in (a,+\infty]$. If $\varphi^{-1}(s) \in [-\infty, b)$, $\varphi^{-1}(s) < b$ $\Rightarrow s < \varphi(b)$. Choose $\varepsilon_b > 0$ so that $|t-s| < \varepsilon_b$ ensures $t < \varphi(b) \Rightarrow \varphi^{-1}(t) \in I_b$. Lastly, if $\varphi^{-1}(s) \in (a,b) = [-\infty, b) \cap (a, +\infty]$, choose $\varepsilon = \min\{\varepsilon_a, \varepsilon_b\}$.

Thus φ^{-1} is continuous.

③ ⓐ Suppose f is lsc, so that $\{x : f(x) \leq a\}$ is closed $\forall a \in \overline{\mathbb{R}}$. Fix $x_0 \in X$ and $x_n \to x_0$.

Case 1: $\liminf_{k\to\infty} f(x_k) = \sup_{k \in \mathbb{N}} \inf_{n \geq k} f(x_n) \neq -\infty$.
For all $\varepsilon > 0$, $\liminf f(x_k) < \liminf f(x_k) + \varepsilon$, so we must have $\inf_{n \geq k} f(x_n) < \liminf f(x_k) + \varepsilon$ for all $k \in \mathbb{N}$. Thus,
$x_n \in \{x : f(x) \leq \liminf_{k\to\infty} f(x_k) + \varepsilon\}$ for infinitely many n. As the set is closed, this gives $x_0 \in \{x : f(x) \leq \liminf_{k\to\infty} f(x_k) + \varepsilon\}$ that is $f(x_0) \leq \liminf_{k\to\infty} f(x_k) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this gives the result.

Case 2: $\liminf_{k\to\infty} f(x_k) = -\infty$, so $\inf_{n \geq k} f(x_n) = -\infty$ $\forall k \in \mathbb{N}$.
For all $M \in \mathbb{R}$, $x_n \in \{x : f(x) \leq M\}$ for infinitely many n, so $x_0 \in \{x : f(x) \leq M\}$ that is $f(x_0) \leq M$. Sending $M \to -\infty$, gives the result.

Conversely, suppose that for all $x_0 \in X$ and $x_n \to x_0$, $\liminf_{n\to\infty} f(x_n) \geq f(x_0)$. For arbitrary $a \in \overline{\mathbb{R}}$, if $\{x : f(x) \leq a\} = \emptyset$, then $\{x : f(x) > a\} = X$ is open. Alternatively, if $\{x : f(x) \leq a\} \neq \emptyset$, let $\{x_n\}_{n=1}^{\infty} \subseteq \{x : f(x) \leq a\}$ be an arbitrary convergent sequence, $x_n \to x_0$. Since $\liminf_{n\to\infty} f(x_n) \geq f(x_0)$, $x_0 \in \{x : f(x) \leq a\}$. Thus, $\{x : f(x) \leq a\}$ is closed, so $\{x : f(x) > a\}$ is open.

(b)

Note that f is upper semicts $\iff$ $\{x : f(x) < a\} = \{x : -f(x) > -a\}$ is open for all $a \in \mathbb{R}$ $\iff$ $-f$ is lower semicts.

By part (a), $-f$ is lower semicts,
$\iff \liminf_{n\to\infty} -f(x_n) \geq -f(x_0) \iff -\liminf_{n\to\infty} -f(x_n) \leq f(x_0)$
$\overset{(2)(a)}{\iff} \limsup_{n\to\infty} f(x_n) \leq f(x_0)$.

(c) Suppose f is continuous. Then, for every convergent sequence x_n, the discussion at the end of (2),

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n).$$

By parts (a) and (b), this shows f is upper and lower semicontinuous.

Suppose f is upper and lower semicontinuous. If x_n is a convergent sequence with limit x_0,

$$\limsup f(x_n) \overset{(b)}{\leq} f(x_0) \overset{(a)}{\leq} \liminf f(x_n).$$

By Q2 (b), this shows $\lim_{n\to\infty} f(x_n) = f(x_0)$. Since x_n was arbitrary, this shows f is continuous.

(4)

(a) It suffices to show the upper and lower Riemann integrals are not equal. Since $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are dense in $\mathbb{R}$, for any partition P and all $i=1,\dots,n$, there exists $q_i \in \mathbb{Q}$ and $p_i \in \mathbb{R}\setminus\mathbb{Q}$ so that $x_{i-1} \leq q_i \leq x_i$ and $x_{i-1} \leq p_i \leq x_i$. Thus, $M_i = 1$ and $m_i = 0$ for all $i=1,\dots,n$. Consequently, $U(P,f) = 1$ and $L(P,f) = 0$ for all partitions P. Therefore,
$\overline{\int_a^b} f(x)\,dx = 1 \neq 0 = \underline{\int_a^b} f(x)\,dx$, which completes the proof.

⑥

ⓐ Fix $x \in \mathbb{R}$. Then, $\exists\, n \in \mathbb{Z}$ s.t. $x \in [n, n+1]$. Thus $x - n \in [0,1]$. Since $n \in \mathbb{Q}$, this shows that the equivalence class $[x]$ contains an element in $[0,1]$.

ⓑ Take $q_1, q_2 \in \mathbb{Q} \cap [-1,1]$, and assume that $A+q_1 \cap A+q_2 \neq \emptyset$. Then $\exists\, x$ s.t. $x = a_1 + q_1$ and $x = a_2 + q_2$ for $a_1, a_2 \in A$. Thus, $a_1 - a_2 = q_2 - q_1$, so $a_1 \sim a_2$. By construction of A, this implies $a_1 = a_2$. Therefore $q_1 = q_2$. This shows $A+q_1$ and $A+q_2$ have nontrivial intersection only if $q_1 = q_2$. Therefore $\{A+q\}_{q \in \mathbb{Q} \cap [-1,1]}$ is a disjoint collection of sets.

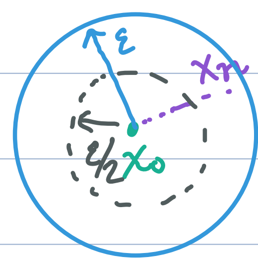
⑦

ⓐ $\varepsilon \mapsto \inf\{f(y) : d(x,y) < \varepsilon\}$ is decreasing

(b) Fix $a \in \overline{\mathbb{R}}$. We must show $\{x : f_*(x) \leq a\}$ is closed. Since $\varepsilon \mapsto \inf\{f(y) : d(x,y) < \varepsilon\}$ is decreasing, $f_*(x) \leq a$ iff, for all $\varepsilon > 0$, $\inf\{f(y) : d(x,y) < \varepsilon\} \leq a$.

Suppose $\{x_n\}_{n=1}^{\infty} \subseteq \{x : f_*(x) \leq a\}$ and $x_n \to x_0$. Then, $\forall \varepsilon > 0$, $\forall n \in \mathbb{N}$, $\inf\{f(y) : d(x_n,y) < \varepsilon\} \leq a$.

Fix $\tilde{\varepsilon} > 0$ arbitrary. It suffices to show $\inf\{f(y) : d(x_0,y) < \tilde{\varepsilon}\} \leq a$.



Choose N so that $d(x_n,x_0) < \frac{\tilde{\varepsilon}}{2}$ $\forall n > N$. Then, $d(x_0,y) < d(x_0,x_n) + d(x_n,y)$, so for $n > N$,

$$\{y : d(x_0,y) < \tilde{\varepsilon}\} \supseteq \{y : d(x_n,y) < \tfrac{\tilde{\varepsilon}}{2}\}.$$

Thus, for $n > N$,

$$\inf\{f(y) : d(x_0,y) < \tilde{\varepsilon}\} \leq \inf\{f(y) : d(x_n,y) < \tfrac{\tilde{\varepsilon}}{2}\} \leq a.$$

This gives the result.

(c) Since $d(x,x)=0<\varepsilon$ for all $\varepsilon>0$, the definition of f_* ensures that $\forall x\in X$
$$f_*(x)=\lim_{\varepsilon\to 0}\left(\inf\{f(y): d(x,y)<\varepsilon\}\right)\le \lim_{\varepsilon\to 0} f(x)=f(x).$$

(d) First, note that
$$\inf\{\liminf_{n\to\infty} f(x_n): x_n\to x\}$$
$$\ge \inf\{\liminf_{n\to\infty} f_*(x_n): x_n\to x\} \quad \text{part (c)}$$
$$\ge \inf\{f_*(x): x_n\to x\} \quad \text{part (a)}$$
$$= f_*(x)$$

To see the other inequality, note that by definition of $f_*(x_0)$, $\forall n\in\mathbb{N}$,

$$f_*(x)\ge \inf\{f(y): d(x_0,y)<\tfrac{1}{n}\}$$

Next, by definition of the infimum, $\forall n\in\mathbb{N}$, $\exists\, x_n\in X$ s.t. $d(x_0,x_n)<\frac{1}{n}$ and $f_*(x)+\frac{1}{n}\ge f(x_n)$.

By construction, $x_n\to x_0$ and $f_*(x)\ge \liminf_{n\to\infty} f(x_n)$.

This shows $f_*(x) \geq \inf\{\liminf_{n\to\infty} f(x_n) : x_n \to x\}$, which completes the proof.

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(a) $f^{-1}(\bigcup_{\alpha\in A} E_\alpha)$
$= \{x \in X : f(x) \in \bigcup_{\alpha\in A} E_\alpha\}$
$= \bigcup_{\alpha\in A} \{x \in X : f(x) \in E_\alpha\}$
$= \bigcup_{\alpha\in A} f^{-1}(E_\alpha)$

(b) $f^{-1}(E^c)$
$= \{x \in X : f(x) \in E^c\}$
$= \{x \in X : f(x) \in E\}^c$
$= (f^{-1}(E))^c$

(c) The first is always true, since
$f(\bigcup_{\alpha\in A} E_\alpha)$
$= \{f(x) : x \in \bigcup_{\alpha\in A} E_\alpha\}$
$= \bigcup_{\alpha\in A} \{f(x) : x \in E_\alpha\}$
$= \bigcup_{\alpha\in A} f(E_\alpha)$.

The second is not always true. Suppose $X = Y = \{0,1\}$ and $f(x) = 0$ $\forall x \in X$. Then $f(\{0\}^c) = f(\{1\}) = \{0\} \neq \{1\} = \{0\}^c = f(\{0\})^c$.