Math 201a: Homework 2

Due Sunday, October 13th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice questions will appear on each exam.

Question 1

Prove the following:

(a) Any algebra that is closed under countable disjoint unions is a σ-algebra.

(b) Any algebra that is closed under countable increasing unions is a σ-algebra. (We say that an algebra $\mathcal{A}$ is closed under countable increasing unions if, for all $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ with $E_i \subseteq E_{i+1}$ for all i, $\cup_{i=1}^{\infty} E_i \in \mathcal{A}$.)

Question 2

(a) Prove the following lemma from lecture 2:

LEMMA 1. *Suppose $\mathcal{A}$ is an algebra of subsets of a set X.*

(i) If $E_1, \dots, E_n \in \mathcal{A}$, then $\cap_{i=1}^{n} E_i \in \mathcal{A}$.
(ii) $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$.

(b) Prove the following claim from lecture 2:

Claim: *Given a nonempty collection $\mathcal{C}$ of σ-algebras on X,*

$$\cap \mathcal{C} := \{E : E \in \mathcal{A}, \ \forall \mathcal{A} \in \mathcal{C}\}$$

is a σ-algebra.

Question 3

Fix an open set $U \subseteq \mathbb{R}$. Prove that there exist $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ so that $U = \cup_{n=1}^{\infty} (a_n, b_n)$.

Question 4*

Prove the following proposition from lecture 2:

PROPOSITION 2. *The Borel σ-algebra of $\mathbb{R}$ is generated by each of the following:*

(i) the half-open intervals: $\mathcal{E}_3 = \{(a, b] : a < b\}$,

(ii) the open rays: $\mathcal{E}_5 = \{(a, +\infty) : a \in \mathbb{R}\}$,

(iii) the closed rays: $\mathcal{E}_7 = \{[a, +\infty) : a \in \mathbb{R}\}$.

Question 5*

Let X be an uncountable set, and let $\mathcal{A}$ be the collection of subsets $E \subseteq X$ such that either E or E^c is at most countably infinite. Prove that $\mathcal{A}$ is a σ-algebra.

Question 6*

Given a collection of sets $\{E_i\}_{i=1}^{\infty}$, we may define

$$\limsup_{i\to+\infty} E_i := \cap_{k=1}^{\infty} \cup_{i=k}^{\infty} E_i \ , \qquad \liminf_{i\to+\infty} E_i := \cup_{k=1}^{\infty} \cap_{i=k}^{\infty} E_i \ .$$

We may also consider the sets

$$A_1 := \{x : x \in E_i \text{ for all but finitely many } i\} \ , \qquad A_2 := \{x : x \in E_i \text{ for infinitely many } i\} \ .$$

Determine for which values of $n, m \in \{1, 2\}$, $\limsup_{i\to+\infty} E_i = A_n$ and $\liminf_{i\to+\infty} E_i = A_m$. Justify your answer with a proof.

Question 7

Each $x \in [0, 1]$ has a base-10 decimal expansion $x = \sum_{j=1}^{+\infty} a_j 10^{-j}$, where $a_j = 0, 1, \ldots, 9$. This expansion is unique unless x is of the form $p10^{-k}$, for $p, k \in \mathbb{N}$, where p is not divisible by 10. In this case x has two expansions: one with $a_j = 0$ for $j > k$ and one with $a_j = 9$ for $j > k$. For the purposes of this problem, suppose that we always choose the *standard decimal representation of* x, which is the one for which $a_j = 0$ for $j > k$.

(a) Fix $n \in \mathbb{N}$. Let A_n be the set of numbers x in the interval $[0, 1]$ with a 7 in the nth decimal place. Prove that A_n belongs to the Borel σ-algebra.

(b) Let E be the set of numbers x in the interval $[0, 1]$ so that the decimal expansion for x contains a 7. Prove that E belongs to the Borel σ-algebra.

(c) Let F be the set of numbers x in the interval $[0, 1]$ so that the decimal expansion for x contains infinitely many 7's. Prove that F belongs to the Borel σ-algebra.

Question 8*

Consider a metric space (X, d) and $f : X \to \overline{\mathbb{R}}$. Recall the notion of lower semicontinuous envelope f_* from Homework 1. Let f^* denote the upper semicontinuous envelope of f, which is defined symmetrically. You may use all the analogous properties of the upper semicontinuous envelope f^* that you proved for the lower semicontinuous envelope f_* on Homework 1.

(a) Prove that f is continuous at $x \in X$ if and only if $f_*(x) = f^*(x) = f(x)$.

(b) Let E be the set of points at which f is discontinuous. Prove that E is a countable union of closed sets, hence a Borel set. (*Hint*: What can you say about the set $\{y : f_*(y) \geq f^*(y)\}$?)

Question 9*

Suppose $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) < +\infty$. Suppose $A_1, A_2, \ldots$ are sets in $\mathcal{M}$ with $\mu(A_i) \geq c > 0$ for all i. Let Z be the set of elements $x \in X$ that belong to infinitely many of the A_i's. Prove that $\mu(Z) \geq c$.

Question 10

Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$ nonempty, define *the restriction of* μ *to* E by $\mu_E(A) := \mu(A \cap E)$ for all $A \in \mathcal{M}$. Prove that μ_E is a measure on $\mathcal{M}$.

Question 11*

Suppose that X is an uncountable set. Define outer measures μ^* and ν^* on X by

$$\mu^*(E) = \begin{cases} 0 & \text{if } E \text{ is countable,} \\ 1 & \text{if } E \text{ is uncountable,} \end{cases} \qquad \nu^*(E) = \begin{cases} 0 & \text{if } E \text{ is countable,} \\ +\infty & \text{if } E \text{ is uncountable.} \end{cases}$$

(a) Find all sets that are μ^* measurable. Justify your answer.

(b) Find all sets that are ν^* measurable. Justify your answer.

Question 12*

By Caratheodory's Theorem, we know that any outer measure μ^* can lead to a measure by restricting μ^* to the collection of μ^*-measurable sets. However, in this problem, we will show that the collection of μ^*-measurable sets is, in general, not the largest σ-algebra on which the restriction of μ^* becomes a measure.
Consider the set $X = \{1, 2, 3\}$. Define an outer measure as follows:

$$\mu^*(A) = \begin{cases} 0 & \text{if } |A| = 0, \\ 1 & \text{if } |A| = 1, 2 \\ 2 & \text{if } |A| = 3. \end{cases}$$

(a) Prove that μ^* is an outer measure on X.

(b) Prove that the collection of μ^* measurable sets is $\{\emptyset, X\}$.

(c) Prove that $\mathcal{A} := \{\emptyset, \{1\}, \{2, 3\}, X\}$ is a σ-algebra.

(d) Prove that $\mu^*|_{\mathcal{A}}$ is a measure.

This shows that the Caratheodory σ-algebra $\mathcal{M}_{\mu^*}$ is not, in general, the largest σ-algebra on which μ^* can be restricted to be a measure.