Homework 2 Solutions

Math 201a, F24

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(3) Let a_n and b_n be the sequences of all rational numbers so that $(a_n, b_n) \subseteq \mathcal{U}$. Then $\bigcup_{n=1}^{\infty} (a_n, b_n) \subseteq \mathcal{U}$.

To see the opposite containment, fix $u \in \mathcal{U}$. Since $\mathcal{U}$ is open, $\exists\, \varepsilon > 0$ s.t. $(u-\varepsilon, u+\varepsilon) \subseteq \mathcal{U}$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a_n, b_n so that $u \in (a_n, b_n) \subseteq (u-\varepsilon, u+\varepsilon)$. Thus $\mathcal{U} \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$.

(4)

(*)

Recall from Q3 that we may express any open set $\mathcal{U} \subseteq \mathbb{R}$ as $\mathcal{U} = \bigcup_{n=1}^{\infty} (a_n, b_n)$ for some $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$.

(i) Since $(a,b] = \bigcap_{n=1}^{\infty} (a, b+\frac{1}{n})$, it is clear that $\mathcal{E}_3 \subseteq \mathcal{B}_{\mathbb{R}}$, so $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{B}_{\mathbb{R}}$. OTOH $(a,b) = \bigcup_{n=1}^{\infty} (a, b-\frac{1}{n}]$, so $(a,b) \subseteq \mathcal{M}(\mathcal{E}_3)$

and (*) ensures $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{E}_3)$.

(ii) Since $(a,b] = (a,+\infty) \setminus (b,+\infty)$, $\mathcal{E}_3 \subseteq \mathcal{M}(\mathcal{E}_5)$
so part (i) ensures $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_5)$
Since $(a,+\infty) = \bigcup_{n=1}^{\infty} (a, a+n)$, $\mathcal{E}_5 \subseteq \mathcal{B}_{\mathbb{R}}$, so
$\mathcal{M}(\mathcal{E}_5) \subseteq \mathcal{B}_{\mathbb{R}}$.

(iii) Since $[a,+\infty) = \bigcap_{n=1}^{\infty} (a-\frac{1}{n}, +\infty)$, part (ii) ensures
$\mathcal{E}_7 \subseteq \mathcal{M}(\mathcal{E}_5) = \mathcal{B}_{\mathbb{R}}$; so $\mathcal{M}(\mathcal{E}_6) \subseteq \mathcal{B}_{\mathbb{R}}$.

Since $(a,+\infty) = \bigcup_{n=1}^{\infty} [a+\frac{1}{n}, +\infty) \subseteq \mathcal{M}(\mathcal{E}_7)$, part (ii)

ensures $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{E}_6)$.

(5) By defn, $\mathcal{A}$ is nonempty.
First, we show closure under complements.
Suppose $E \in \mathcal{A}$. Then either E or E^c
is at most countably infinite $\iff$
either E^c or $(E^c)^c$ is at most
countably infinite $\iff E^c \in \mathcal{A}$.

Next, we show closure under countable unions. Suppose $E_1, E_2, \ldots \in \mathcal{A}$. Consider $\bigcup_{n=1}^{\infty} E_n$. If $\bigcup_{n=1}^{\infty} E_n$ is at most countably infinite, $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$. On the other hand, suppose $\bigcup_{n=1}^{\infty} E_n$ is <u>not</u> at most countably infinite. Then there exists at least one set E_m in the sequence that is not countable. By defn of $\mathcal{A}$, E_m^c is at most countably infinite. Thus, $\left(\bigcup_{n=1}^{\infty} E_n\right)^c = \bigcap_{n=1}^{\infty} E_n^c \subseteq E_m^c$ is at most countably infinite, so $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$.

⑥ First, we show $\limsup E_i = A_2$. Note that

$x \in A_2 \iff x \in E_i$ for infinitely many i

$\iff \forall k \in \mathbb{N}, \exists i \geq k$ s.t. $x \in E_i$

$\iff \forall k \in \mathbb{N}, x \in \bigcup_{i=k}^{\infty} E_i$

$\iff x \in \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} E_i$

Next, we show $\liminf E_i = A_1$. Note that

$x \in A_1 \iff x \in E_i$ for all but finitely many i

$\iff \exists k \in \mathbb{N}$ s.t. $\forall i \geq k, x \in E_i$

$\iff \exists k \in \mathbb{N}$ s.t. $x \in \bigcap_{i=k}^{\infty} E_i$

$\iff x \in \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} E_i$

(8)

Suppose f is continuous at $x \in X$. Then, for all sequences $x_n \to x$, $\lim_{n\to\infty} f(x_n) = f(x)$.

By HW1, Q7(d), $f_*(x) = \inf\{\liminf f(x_n) : x_n \to x\}$
$$= \inf\{f(x)\}$$
$$= f(x).$$

Similarly, $f^*(x) = \sup\{\limsup f(x_n) : x_n \to x\} = f(x)$.

Now, suppose $f_*(x) = f^*(x) = f(x)$. Then

$$
\begin{aligned}
f_*(x) &= \inf\{\liminf f(x_n) : x_n \to x\} \\
&\le \inf\{\limsup f(x_n) : x_n \to x\} \\
&\le \sup\{\limsup f(x_n) : x_n \to x\} \\
&= f(x) \\
&= f_*(x)
\end{aligned}
$$

Thus, equality holds throughout and, $\forall x_n \to x$, $\limsup f(x_n) = f(x)$. Similarly, $\forall x_n \to x$, $\liminf f(x_n) = f(x)$. Therefore, by HW1, Q1(a), $\lim_{n\to\infty} f(x_n) = f(x)$. This shows f is continuous at x.

(b)

$$E = \{x : f_*(x) < f^*(x)\}$$

$$= \bigcup_{\substack{p,q \in \mathbb{Q} \\ p<q}} \{x : f_*(x) \le p\} \cup \{x : q \le f^*(x)\}$$

since f_* is lsc, this is closed; since f^* is usc, this is open

$$= \bigcup_{\substack{p,q \in \mathbb{Q} \\ p<q}} \overbrace{f_*^{-1}((-\infty, p])}^{\text{since } f_* \text{ is lsc, this is closed}} \cup \overbrace{(f^*)^{-1}([q, +\infty))}^{\text{since } f^* \text{ is usc, this is open}}$$

This is a countable union of closed sets.

(a)

By Q6,

$$Z = \bigcap_{n=1}^{\infty} \bigcup_{k \ge n} A_k$$

Define $B_n = \bigcup_{k \ge n} A_k$. Then $B_1 \supseteq B_2 \supseteq \ldots$

and $\mu(B_1) \le \mu(X) < +\infty$. Thus,

$$\mu(Z) = \mu\left(\bigcap_{n=1}^{\infty} B_n\right) \overset{\text{cty from above}}{=} \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu\left(\bigcup_{k \ge n} A_k\right)$$

$$\ge \lim_{n\to\infty} \mu(A_n) \ge c$$

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(a)

Claim: $\mathcal{M} := \{A \subseteq X : \text{either } A \text{ or } A^c \text{ is countable}\}$ is the collection of μ^*-measurable sets.

Assume $A \in \mathcal{M}$. If A is countable, $\mu^*(A) = 0$, hence A is μ^*-measurable. If A^c is countable, then the preceding sentence shows A^c is μ^*-measurable, and since the collection of μ^*-measurable sets is a σ-algebra, A must also be μ^*-measurable.

Now, suppose $A \notin \mathcal{M}$. Then neither A nor A^c is countable. Consequently,
$\mu^*(X) = 1 \neq 2 = \mu^*(X \cap A) + \mu^*(X \cap A^c)$.
Thus A is not μ^*-measurable.

(b)

Claim $\mathcal{M} := 2^X$ is the collection of ν^*-measurable sets

Fix $A \subseteq X$.

If E is countable, so are $E \cap A$ and $E \cap A^c$, hence $\nu^*(E) = 0 = \nu^*(E \cap A) + \nu^*(E \cap A^c)$.

If E is uncountable, then either $E \cap A$ or $E \cap A^c$ is uncountable. Thus, $\nu^*(E) = +\infty = \nu^*(E \cap A) + \nu^*(E \cap A^c)$.

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(a) By definition $\mu^*(\emptyset) = 0$ and $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$. Given $\{A_i\}_{i=1}^{\infty} \subseteq 2^X$, first, suppose $|\bigcup_{i=1}^{\infty} A_i| = 0$. Then $|A_i| = 0\ \forall i$, so

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = 0 = \sum_{i=1}^{\infty} \mu^*(A_i).$$

Next, suppose $|\bigcup_{i=1}^{\infty} A_i| = 1$. Then there must be some i s.t. $|A_i| = 1$. Thus

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = 1 \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

Next, suppose $|\bigcup_{i=1}^{\infty} A_i| = 2$. Then there must either be $i \neq j$ s.t. $|A_i| = |A_j| = 1$ or i s.t. $|A_i| = 2$. In both cases,

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = 1 \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

Lastly, suppose $|\bigcup_{i=1}^{\infty} A_i| = 3$.
Then there must either be i, j, k distinct with $|A_i| = 1$ or i and j distinct with $|A_i| = 2$, $|A_j| = 1$ or k s.t. $|A_k| = 3$. In all cases,

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = 2 \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

(b) Suppose $A \subseteq X$ has one element. WLOG, $A = \{1\}$. Then,

$$\mu^*(\{1,2\}) = 1 \neq 1 + 1 = \mu^*(\{1,2\} \cap A) + \mu^*(\{1,2\} \cap A^c)$$

Thus A and A^c are not μ^*-meas. This shows $\mathcal{M}_{\mu^*} = \{\emptyset, X\}$.

(c) $\mathcal{A}$ is closed under complements and arbitrary unions. Thus, it is a σ-algebra.

(d) It suffices to show $\mu^*|_{\mathcal{A}}$ is countably additive. The only nontrivial disjoint unions are of the form $\{1\} \cup \{2,3\}$. In this case,

$$\mu^*|_{\mathcal{A}}(\{1\} \cup \{2,3\}) = 2 = 1 + 1 = \mu^*|_{\mathcal{A}}(\{1\}) + \mu^*|_{\mathcal{A}}(\{2,3\}).$$

[illegible]