Math 201a: Homework 3

Due Sunday, October 20th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice questions will appear on each exam. *All answers should be justified with either a proof or a counterexample.*

Question 1

Suppose $\lambda^*(A) > 0$. Prove that, for all $\epsilon > 0$, there exists a nonempty interval $(a, b]$, $a, b \in \mathbb{R}$ so that

$$\lambda^*(A \cap (a, b]) > (1 - \epsilon)\lambda^*((a, b]).$$

Hint: consider the cases $\lambda^*(A) < +\infty$ and $\lambda^*(A) = +\infty$ separately. For the first case estimate $s := \sup_{a<b\in\mathbb{R}} \lambda^*(A \cap (a, b])/\lambda^*((a, b])$—for example, can you show that for any covering $\{(a_i, b_i]\}_{i=1}^{\infty}$ of A, we have $\lambda^*(A) \leq s \sum_{i=1}^{\infty} \lambda^*((a_i, b_i])$? For the second case, approximate A by a set B with $\lambda^*(B) < +\infty$ to reduce to the first case.

Question 2*

In this problem, we will show how any measure can be extended to an outer measure.

Given a measure space $(X, \mathcal{A}, \mu)$, for any $A \subseteq X$ define

$$\mu^*(A) = \inf\{\mu(B) : A \subseteq B \text{ and } B \in \mathcal{A}\}.$$

(a) Prove that the infimum is attained.

(b) Prove that μ^* is an outer measure.

(c) Prove that if $A \in \mathcal{A}$, then A is μ^*-measurable and $\mu^*(A) = \mu(A)$.

(d) The textbook defines the extension of a measure μ to an outer measure in a slightly more general setting, but in the setting of the problem, the extension is given by

$$\tilde{\mu}^*(A) = \inf\left\{\sum_{i=1}^{+\infty} \mu(B_i) : A \subseteq \cup_{i=1}^{+\infty} B_i,\ \{B_i\}_{i=1}^{+\infty} \subseteq \mathcal{A}\right\}.$$

Prove that $\tilde{\mu}^*(A) = \mu^*(A)$ for all $A \subseteq X$, so that the book's extension is equivalent to the extension considered in this problem.

Question 3*

Let μ^* be an outer measure on a nonempty set X. Suppose $A \subseteq X$ is not μ^* measurable and that $\mu^*(A) < +\infty$. Prove that there exists $S \subseteq A$ satisfying

$$\mu^*(S) > 0, \text{ and for all } T \subsetneq S \text{ so that } T \text{ is } \mu^*\text{-measurable},\ \mu^*(T) = 0.$$

Hint: Show that there exists $B \subseteq A$ such that B is the μ^*-measurable subset of A with the largest measure. Then consider the set $A \setminus B$.

Question 4

Let X be any nonempty set. Let $\mathcal{C}$ be a collection of subsets of X such that $\emptyset \in \mathcal{C}$. Let $\varphi : \mathcal{C} \to [0, +\infty]$ be any function satisfying $\varphi(\emptyset) = 0$. Define $\mu^* : 2^X \to [0, +\infty]$ by

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \varphi(B_i) : A \subseteq \cup_{i=1}^{\infty} B_i,\ B_i \in \mathcal{C} \right\}.$$

(a) Prove that μ^* is an outer measure. (If we are taking the infimum of a function over a constraint set that is empty, we adopt the convention that the infimum is $+\infty$.)

(b) Give an example of X, $\mathcal{C}$, and A for which the constraint set in the infimum is empty.

Question 5

Prove that every $E \in \mathcal{M}_{\lambda^*}$ with $\lambda(E) > 0$ contains a subset A so that $A \notin \mathcal{M}_{\lambda^*}$.

Hint: First, reduce to the case $E \subseteq [0, 1]$. Recall the equivalence relation $x \sim y \iff x - y \in \mathbb{Q}$. Justify why, for each equivalence class that contains an element of E, you can choose a representative that belongs to E. Prove that the resulting set cannot be measurable by proceeding by contradiction: suppose it is measurable and show that this implies $\lambda(E) = 0$, similarly to how we proceeded in Lecture 1.

Question 6

In HW2 Q12, you showed that, given an outer measure μ^*, the collection of μ^* measurable sets $\mathcal{M}_{\mu^*}$ is not necessarily the largest σ-algebra on which μ^* can be restricted to be a measure. In this problem, you will show that, as long as the outer measure of any set can be approximated by a μ^*-measurable set containing it, then the collection of μ^* measurable sets *is* maximal.

Let X be a nonempty set and suppose μ^* is an outer measure on X. Suppose that, for all $S \subseteq X$ and for all $\epsilon > 0$, there exists a μ^*-measurable set $E \supseteq S$ so that $\mu^*(E) \leq \mu^*(S) + \epsilon$.

(a) Suppose A is not μ^* measurable and consider the σ-algebra $\mathcal{F}$ generated by $\mathcal{M}_{\mu^*}$ and $\{A\}$. Prove that μ^* is not additive on $\mathcal{F}$.

(b) Use part (a) to conclude that $\mathcal{M}_{\mu^*}$ is the largest σ-algebra on which μ^* can be restricted to be a measure. *(Hint: this is almost immediate from part (a).)*

(c) Consider a measure space $(X, \mathcal{A}, \nu)$. Suppose that μ^* is the outer-measure induced by ν, according to the equation at the beginning of Q2 above. Use the previous parts to prove that the collection of μ^* measurable sets is the largest σ-algebra on which μ^* can be restricted to be a measure.

Later, we will give an example where $\mathcal{A} \neq \mathcal{M}_{\mu^*}$.

Question 7

For any non-decreasing function $F : \mathbb{R} \to \mathbb{R}$, the left and right limits exist at any $a \in \mathbb{R}$, and we will abbreviate them in the following way:

$$F(a+) = \lim_{x \to a^+} F(x), \quad F(a-) = \lim_{x \to a^-} F(x).$$

Now, suppose $F : \mathbb{R} \to \mathbb{R}$ is also right continuous, that is $F(a+) = F(a)$, for all $a \in \mathbb{R}$. Let μ_F be the associated Lebesgue-Stieltjes measure. Prove the following: for any $a, b \in \mathbb{R}$,

(i) $\mu_F(\{a\}) = F(a) - F(a-)$;

(ii) $\mu_F([a, b)) = F(b-) - F(a-)$;

(iii) $\mu_F([a, b]) = F(b) - F(a-)$;

(iv) $\mu_F((a, b)) = F(b-) - F(a)$.

Question 8*

In Lecture 1, we defined (using the Axiom of Choice) a set $A \subseteq [0, 1]$ with the following property: for every $x \in \mathbb{R}$ there is exactly one $y \in A$ such that $x - y \in \mathbb{Q}$.

(a) Prove that $\lambda^*(A) > 0$. *Hint: use our argument from class*

(b) Prove that if $S \subseteq A$ satisfies $S \in \mathcal{M}_{\lambda^*}$, then $\lambda(S) = 0$. *Hint: Consider $S + 1/n$ for $n \in \mathbb{N}$.*

(c) Use parts (a) and (b) to conclude that A is not Lebesgue measurable. *Hint: this is almost immediate.*

Question 9*

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\mathcal{N}$ denote the null sets of μ, that is

$$\mathcal{N} = \{N \in \mathcal{A} : \mu(N) = 0\}.$$

A measure whose domain includes all subsets of null sets is called *complete*.

In Theorem 1.9 of Folland, it is shown that the *completion* of a σ-algebra $\mathcal{A}$, defined by

$$\overline{\mathcal{A}} := \{A \cup B : A \in \mathcal{A} \text{ and } B \subseteq N \text{ for some } N \in \mathcal{N}\},$$

is a σ-algebra and there is a unique extension of μ to $\overline{\mathcal{A}}$, given by

$$\bar{\mu}(A \cup B) := \mu(A),$$

which is known as the *completion* of μ. In particular, it is clear from the definition that $\bar{\mu}$ defined on $\bar{\mathcal{A}}$ is a complete measure.

In this problem, we will show that Lebesgue measure, defined on the collection of Lebesgue measurable sets, is the completion of Lebesgue measure, defined on the Borel σ-algebra. Throughout, given a measure space $(X, \mathcal{A}, \mu)$, we will let μ^* be the outer measure induced by μ, as in Q2, and let $\mathcal{M}_{\mu^*}$ be the associated collection of μ^* measurable sets.

(a) Suppose μ is *σ-finite*. Prove $E \in \mathcal{M}_{\mu^*}$ if and only if there exists $A \in \mathcal{A}$ so that $E \subseteq A$ and $\mu^*(A \setminus E) = 0$.

(b) Suppose μ is *σ-finite*. Prove that $\mathcal{M}_{\mu^*} = \{A \cup B : A \in \mathcal{A} \text{ and } B \subseteq N \text{ for some } N \in \mathcal{N}\}$. In other words, $\mathcal{M}_{\mu^*}$ is the completion of $\mathcal{A}$.

(c) Suppose $\mu = \lambda$ and $\mathcal{A} = \mathcal{B}_{\mathbb{R}}$. Prove that the outer measure induced by μ coincides with our definition of Lebesgue outer measure from class, μ_F^* for $F(x) = x$. Prove that the completion of μ coincides with the restriction of Lebesgue outer measure to Lebesgue measurable sets.

On a subsequent problem set, you will show that Lebesgue measure on the Borel sets is *not* complete.

Question 10*

(i) Prove that $\mathcal{B}_{\bar{\mathbb{R}}}$ is generated by sets of the form $(a, +\infty]$ for $a \in \mathbb{R}$.

(ii) Prove that $\mathcal{B}_{\bar{\mathbb{R}}} = \{E \subseteq \bar{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$.