Homework 3 Solutions

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ⓐ First, suppose $\mu^*(A) = +\infty$. Then we must have $\mu(B) = +\infty$ for all B s.t. $A \subseteq B$, $B \in \mathcal{A}$. In particular, $B = X$ attains the infimum.

Now, suppose $\mu^*(A) < +\infty$.
Take a minimizing sequence, that is a sequence B_n s.t. $A \subseteq B_n$, $B_n \in \mathcal{A}$ and $\lim_{n\to\infty} \mu(B_n) = \mu^*(A)$. WLOG, we may assume $\mu(B_n) < +\infty$ $\forall n \in \mathbb{N}$.

Define $C = \bigcap_{n=1}^{\infty} B_n$. Then $C \subseteq B_n$ $\forall n$, $C \in \mathcal{A}$ and $A \subseteq C$. Thus,

$$\mu^*(A) = \lim_{n\to\infty} \mu(B_n) \overset{C \subseteq B_n}{\geq} \mu(C) \geq \mu^*(A)$$

Thus, equality must hold throughout, so C attains the minimum.

(b) To see $\mu^*(\emptyset)=0$, take $B=\emptyset$ in the definition of μ^*.

If $E\subseteq F$, then the **constraint** set for F (i.e. the collection of sets $\{B_i\}_{i=1}^{+\infty}\subseteq\mathcal{A}$ s.t. $F\subseteq\bigcup_i B_i$) is contained in the **constraint** set for E. Thus, the inf over the larger set is smaller: $\mu^*(E)\leq\mu^*(F)$.

Fix $\{A_i\}_{i=1}^{+\infty}\subseteq 2^{X}$. Choose $\{B_i\}_{i=1}^{+\infty}\subseteq\mathcal{A}$ that attain the minimum in the defn of $\mu^*(A_i)$.

Then, since $\bigcup_{i=1}^{+\infty}A_i\subseteq\bigcup_{i=1}^{+\infty}B_i$, $\bigcup_{i=1}^{+\infty}B_i\in\mathcal{A}$,

$$\mu^*\Big(\bigcup_{i=1}^{+\infty}A_i\Big)\leq\mu\Big(\bigcup_{i=1}^{+\infty}B_i\Big)\overset{\text{countable subadditivity of }\mu}{\leq}\sum_{i=1}^{+\infty}\mu(B_i)=\sum_{i=1}^{+\infty}\mu^*(A_i).$$

This shows μ^* is an outer measure.

(c) Suppose $A\in\mathcal{A}$. Fix arbitrary $E\subseteq X$. Choose F that attains the minimum in the defn of $\mu^*(E)$. Then,

$$\mu^*(E)=\mu(F)\overset{\text{countable additivity of }\mu}{=}\underbrace{\mu(F\cap A)}_{\substack{F\cap A\in\mathcal{A}\\ E\cap A\subseteq F\cap A}}+\underbrace{\mu(F\cap A^c)}_{\substack{F\cap A^c\in\mathcal{A}\\ E\cap A^c\subseteq F\cap A^c}}\qquad(*)$$

$$\geq\mu^*(E\cap A)+\mu^*(E\cap A^c),$$

so A is μ^*-measurable. Choosing $E=A$, in equation (*), we obtain

$$\mu^*(A) = \underbrace{\mu(F\cap A)}_{=\mu(A),\ \text{since } E\subseteq F} + \overbrace{\mu(F\cap A^c)}^{\geq 0} \geq \mu(A).$$

The opposite inequality is true by defn of μ^*.

d By definition, $\tilde{\mu}^*(A) \leq \mu^*(A)$, so it suffices to show the opposite inequality.

choose $B_1=B$, $B_n=\emptyset\ \forall n>1$

Fix $A\subseteq X$. Choose $\{B_i\}_{i=1}^{+\infty}\subseteq \mathcal{A}$ s.t. $A\subseteq \bigcup_{i=1}^{+\infty} B_i$. Then,

$$\sum_{i=1}^{+\infty}\mu(B_i) \overset{\text{countable subadditivity}}{\geq} \mu\left(\bigcup_{i=1}^{+\infty} B_i\right) \geq \mu^*(A).$$

Taking the infimum over the left hand side gives the result.

(3)

Define $M := \sup\{\mu^*(B) : B \subseteq A, B\ \mu^*\text{-meas}\}$. $(*)$
Since $\mu^*(A) < +\infty$, monotonicity ensures $M < +\infty$.

Furthermore, since $\mu^*(B) \geq 0\ \forall B$, we have $M \geq 0$.

Let B_n be a maximizing sequence, that is $B_n \subseteq A$, B_n μ^*-meas $\forall n \in \mathbb{N}$ and
$$\lim_{n\to\infty} \mu^*(B_n) = M.$$

Define $B = \bigcup_{n=1}^{\infty} B_n$. Then B is μ^*-measurable $B \subseteq A$, and $B_n \subseteq B$. Thus
$$M \geq \mu^*(B) \geq \mu^*(B_n).$$

Taking limits gives,

$$M \geq \mu^*(B) \geq \lim_{n\to\infty} \mu^*(B_n) = M.$$

Thus, B attains the maximum in $(*)$.

Consider the set $S := A \setminus B$. Since B is μ^*-measurable and $A = S \cup B$, S must not be μ^*-measurable, or else A would be, too. Thus, $\mu^*(S) > 0$.

Suppose $T \subsetneq S$ and T is μ^*-measurable. Then $B \cup T \subseteq A$ and $B \cup T$ is μ^*-measurable, so by definition of M,

$$M \geq \mu^*(B \cup T).$$

On the other hand, by the μ^*-measurability of B,

$$M \geq \mu^*(B \cup T) = \mu^*((B \cup T) \cap B) + \mu^*((B \cup T) \cap B^c)$$

$T \subsetneq S = A \setminus B$

$$= \mu^*(B) + \mu^*(T)$$

$$= M + \mu^*(T).$$

Thus, $\mu^*(T) = 0$.

⑧

CLAIM: If $p,q \in \mathbb{Q}$, $p \neq q$, then $A+p \cap A+q = \emptyset$

Pf of CLAIM: Suppose $\exists\, x \in A+p \cap A+q$. Then $x = a_1 - p = a_2 - q$ for $a_1, a_2 \in A$, $a_1 \neq a_2$. This contradicts the definition of A as a set with the property that for every $x \in \mathbb{R}$, there is exactly one $y \in A$ s.t. $x-y \in \mathbb{Q}$.

ⓐ

By CLAIM, $\bigcup_{q\in\mathbb{Q}} A+q$ is a disjoint union and

$(0,1] \subseteq \bigcup_{q\in\mathbb{Q}} A+q$

By countable ~~sub~~additivity, monotonicity, and translation invariance of Lebesgue outer measure,

$$1 = \lambda^*((0,1]) \leq \lambda^*\Big(\bigcup_{q\in\mathbb{Q}} A+q\Big) = \sum_{q\in\mathbb{Q}} \lambda^*(A+q) = \sum_{q\in\mathbb{Q}} \lambda^*(A)$$

Thus, $\lambda^*(A) > 0$.

ⓑ

Suppose $S \in \mathcal{M}_{\lambda^*}$ and $S \subseteq A$. Then the CLAIM ensures $\bigcup_{n=1}^{\infty} S+\frac{1}{n}$ is a disjoint union.

By countable additivity, monotonicity, and translation invariance of Lebesgue measure,

$$\sum_{n=1}^{\infty} \lambda(S) = \sum_{n=1}^{\infty} \lambda\left(S+\tfrac{1}{n}\right) = \lambda\left(\bigcup_{n=1}^{\infty} S+\tfrac{1}{n}\right) \leq \lambda((-1,2]) = 3.$$

$S+\frac{1}{n} \subseteq A+\frac{1}{n} \subseteq [0,1]+\frac{1}{n}$

Thus, $\lambda(S)=0$.

(c) Suppose A were Lebesgue measurable. Then part (b) ensures $\lambda^*(A)=0$. This is a contradiction.

(9)

(a) WTS $E \in \mathcal{M}_{\mu^*} \iff \exists\, A \in \mathcal{A}$ s.t. $E \subseteq A$, $\mu^*(A \setminus E) = 0$.

First, we show "$\Leftarrow$". Fix $S \subseteq X$. Then, for all $\varepsilon > 0$, $\exists\ B_\varepsilon \in \mathcal{A}$ s.t. $S \subseteq B_\varepsilon$ and

$$\mu^*(S) \geq \mu(B_\varepsilon) - \varepsilon$$

$$\overset{\text{additivity of } \mu}{=} \mu(B_\varepsilon \cap A) + \mu(B_\varepsilon \cap A^c) - \varepsilon$$

$$= \mu^*(B_\varepsilon \cap A) + \mu^*(B_\varepsilon \cap A^c) - \varepsilon$$

$$\overset{\text{monotonicity}}{\geq} \mu^*(S \cap E) + \mu^*(S \cap A^c) - \varepsilon$$

$$\geq \mu^*(S \cap E) + \mu^*(S \cap E^c) - \varepsilon. \qquad (*)$$

where the last inequality follows from the fact that

$$\mu^*(S \cap E^c) \overset{\text{subadd}}{\leq} \mu^*(S \cap A^c) + \mu^*(S \cap E^c \cap A)$$

$$\leq \mu^*(S \cap A^c) + \underbrace{\mu^*(A \setminus E)}_{0 \text{ by assumption}}.$$

Sending $\varepsilon \to 0$ in (★) shows $E \in \mathcal{M}_{\mu^*}$.

Now, we show "$\Rightarrow$". Choose B that attains the infimum in the definition of the outer measure (see HW3, Q2(a)), so

$$E \subseteq B, \quad B \in \mathcal{A}, \quad \mu^*(E) = \mu(B).$$

Then,

$$\mu(B) \overset{\text{since } E \in \mathcal{M}_{\mu^*}}{=} \mu^*(B \cap E) + \mu^*(B \cap E^c)$$

$$= \mu^*(E) + \mu^*(B \setminus E).$$

If $\mu^*(E) < +\infty$, this gives the result.

Finally, suppose $\mu^*(E) = +\infty$. Since μ is σ-finite, there exist $\{E_i\}_{i=1}^{\infty} \in \mathcal{A}$ s.t. $\bigcup_i E_i = X$ and $\mu(E_i) < +\infty$.

By monotonicity, $\mu^*(E \cap E_i) \le \mu^*(E_i) = \mu(E_i) < \infty$, and since $\mathcal{A} \subseteq \mathfrak{M}_{\mu^*}$ (HW3, Q2(c)), $E \cap E_i \in \mathfrak{M}_{\mu^*}$. By what we have just shown, $\exists \{A_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ s.t. $E_i \cap E \subseteq A_i$ and

$$0 = \mu^*(A_i \setminus (E_i \cap E)) = \mu^*(A_i \cap (E_i^c \cup E^c)).$$

Let $A = \bigcup_{i=1}^{\infty} A_i$. Then $E \subseteq A$, $A \in \mathcal{A}$ and

$$\mu^*(A \setminus E) = \mu^*(A \cap E^c) \overset{\text{subadd}}{\le} \sum_i \mu^*(A_i \cap E^c)$$

$$\overset{\text{monotonicity}}{\le} \sum_i \mu^*(A_i \cap (E_i^c \cup E^c))$$

$$= 0.$$

$\square$

(b) By part (a), if $E \in \mathfrak{M}_{\mu^*}$, then $E^c \in \mathfrak{M}_{\mu^*}$, so $\exists$ $A \in \mathcal{A}$ s.t. $E^c \subseteq A$ and

$$0 = \mu^*(A \setminus E^c) = \mu^*(A \cap E) = \mu^*(E \setminus A^c).$$

Thus $A^c \subseteq E$ and $E = A^c \cup (E \setminus A^c)$

Furthermore, by HW3, Q2 (b), $\exists$ $N \in \mathcal{A}$ s.t. $E \setminus A^c \subseteq N$ and $\mu(N) = \mu^*(E \setminus A^c) = 0$.

This shows $E \in \{A \cup B : A \in \mathcal{A} \text{ and } B \subseteq N \text{ for } N \in \mathcal{N}\}$.

Conversely, if $E = A \cup B$, for A and B as above, then defining $F := A \cup N$, we see $E \subseteq F$, $F \in \mathcal{A}$ and

$$\mu^*(F \setminus E) = \mu^*((A \cup N) \cap (A^c \cap B^c))$$

$$= \mu^*(N \cap (A^c \cap B^c))$$

$$\leq \mu^*(N)$$

$$= \mu(N)$$

$$= 0,$$

so $E \in \mathcal{M}_{\mu^*}$ by part (a).

(c) By 6.2(d), it is clear that

$$\mu_F^*(A) \geq \mu^*(A),$$

since we may take the B_i to be half open intervals. On the other hand, for any $B \supseteq A$, $B \in \mathcal{B}_\mathbb{R}$,

$$\mu_F^*(A) \leq \mu_F^*(B) = \mu(B).$$

Taking the infimum over all such B gives

$$\mu_F^*(A) \leq \mu^*(A).$$

(10)

(a) Define $\mathcal{F} = \{(a, +\infty] : a \in \mathbb{R}\}$. We seek to show $\mathcal{B}_{\overline{\mathbb{R}}} = \mathcal{M}(\mathcal{F})$.

First, we show $\mathcal{B}_{\overline{\mathbb{R}}} \subseteq \mathcal{M}(\mathcal{F})$. Fix $\mathcal{U} \subseteq \overline{\mathbb{R}}$ open. It suffices to show $\mathcal{U} \in \mathcal{M}(\mathcal{F})$.

Note that $\bigcap_{i=1}^{\infty} (i, +\infty] = \{+\infty\} \in \mathcal{M}(\mathcal{F})$. Likewise, $\left(\bigcup_{i=1}^{\infty} (-i, +\infty]\right)^c = \{-\infty\} \in \mathcal{M}(\mathcal{F})$. Thus, $(a, +\infty) = (a, +\infty] \setminus \{+\infty\} \in \mathcal{M}(\mathcal{F})$. By Q4

$$\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{F}).$$

Since $\{+\infty, -\infty\} = \{x \in \overline{\mathbb{R}} : -\infty < x < +\infty\}^c$ is closed, $\mathcal{U} \setminus \{+\infty, -\infty\}$ is open in $\overline{\mathbb{R}}$, hence open in $\mathbb{R}$. Thus

$$\mathcal{U} \setminus \{+\infty, -\infty\} \in \mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{F}_1)$$

Thus, if $\{+\infty, -\infty\} \cap \mathcal{U} = \emptyset$, we are done.

On the other hand, if either or both are in $\mathcal{U}$ we can use that $\{+\infty\}, \{-\infty\} \in \mathcal{M}(\mathcal{F}_1)$ and that $\mathcal{M}(\mathcal{F}_1)$ is closed under finite unions to obtain

$$\mathcal{U} \in \mathcal{M}(\mathcal{F}_1).$$

Conversely, to show $\mathcal{M}(\mathcal{F}_1) \subseteq \mathcal{B}_{\overline{\mathbb{R}}}$, it suffices to observe that all the sets in $\mathcal{F}_1$ are open.

(b)

First note that $\mathcal{M} := \{E \subseteq \overline{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$ is a σ-algebra:

- If $A \in \mathcal{M}$, then $A^c \cap \mathbb{R} = ((\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}) \setminus A) \cap \mathbb{R}$
  $= \mathbb{R} \setminus A = \mathbb{R} \setminus (A \cap \mathbb{R}) \in \mathcal{B}_{\mathbb{R}}$, since $A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$ and σ-algebras are closed under complements.
- If $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$, then $\left(\bigcup_{i=1}^{\infty} A_i\right) \cap \mathbb{R} = \bigcup_{i=1}^{\infty} (A_i \cap \mathbb{R}) \in \mathcal{B}_{\mathbb{R}}$, since $A_i \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$ $\forall i \in \mathbb{N}$ and σ-algebras are closed under countable unions.

By part (a), $\mathcal{B}_{\overline{\mathbb{R}}}$ is generated by sets of the form $[a,+\infty]$ $a \in \mathbb{R}$.
Since all of these sets belong to $\mathcal{M}$, $\mathcal{B}_{\overline{\mathbb{R}}} \subseteq \mathcal{M}$.

Now, we will show $\mathcal{M} \subseteq \mathcal{B}_{\overline{\mathbb{R}}}$. First note that $\mathcal{A} := \{F \cap \mathbb{R} : F \in \mathcal{B}_{\overline{\mathbb{R}}}\}$ is a σ-algebra on $\mathbb{R}$, since...

- $E \in \mathcal{A} \Rightarrow E = F \cap \mathbb{R}$ for $F \in \mathcal{B}_{\overline{\mathbb{R}}}$
  $\Rightarrow \mathbb{R} \setminus E = \mathbb{R} \cap E^c = \mathbb{R} \cap F^c \in \mathcal{A} \Rightarrow \mathbb{R} \setminus E \in \mathcal{A}$.
- $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A} \Rightarrow E_i = F_i \cap \mathbb{R}$ for $\{F_i\}_{i=1}^{\infty} \subseteq \mathcal{B}_{\overline{\mathbb{R}}}$
  $\Rightarrow \bigcup_{i=1}^{\infty} E_i = \left(\bigcup_{i=1}^{\infty} F_i\right) \cap \mathbb{R} \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$

Furthermore, $\forall a \in \mathbb{R}$, $[a,+\infty) = [a,+\infty] \cap \mathbb{R} \in \mathcal{A}$.
Thus $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{A}$.

Fix $E \in \mathcal{M}$ s.t. $E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$. Then $E \cap \mathbb{R} = F \cap \mathbb{R}$ for some $F \in \mathcal{B}_{\overline{\mathbb{R}}}$. Thus $E = F \cup A$ for $A \in \{\emptyset, \{+\infty\}, \{-\infty\}, \{+\infty, -\infty\}\}$. Since $A \in \mathcal{B}_{\overline{\mathbb{R}}}$, this shows $E \in \mathcal{B}_{\overline{\mathbb{R}}}$. Thus $\mathcal{M} \subseteq \mathcal{B}_{\overline{\mathbb{R}}}$.