Math 201a: Homework 4

Due Sunday, November 3rd at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice questions will appear on each exam. *All answers should be justified with either a proof or a counterexample.*

Question 1

Suppose μ is a *locally finite Borel measure* on $\mathbb{R}$, that is $\mu(B) < +\infty$ for all balls $B \subseteq \mathbb{R}$.

(i) Prove that, for all $S \in \mathcal{B}_{\mathbb{R}}$ and $\epsilon > 0$, there exists an open set U satisfying $S \subseteq U$ so that $\mu(U \setminus S) \leq \epsilon$.

(ii) Prove that, for all $S \in \mathcal{B}_{\mathbb{R}}$ and $\epsilon > 0$ there exists a closed set $K \subseteq S$ so that $\mu(S \setminus K) \leq \epsilon$.

(iii) As a consequence of these results, prove that any locally finite measure is inner and outer regular. (This can also be seen as a consequence of Theorem 1.16 in Folland, which shows all locally finite Borel measures on $\mathbb{R}$ are Lebesgue Stieltjes measures.)

Question 2

Let $(X, \mathcal{M})$ be a measurable space.

(i) Consider $f : X \to \mathbb{R}$. Prove that the following are equivalent:

$$f \text{ is } (\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}}) \text{ measurable} \iff f \text{ is } (\mathcal{M}, \mathcal{B}_{\mathbb{R}}) \text{ measurable.}$$

(ii) Consider $f : X \to \overline{\mathbb{R}}$. Prove that f is measurable if and only if $f^{-1}(\{-\infty\}) \in \mathcal{M}$, $f^{-1}(\{+\infty\}) \in \mathcal{M}$, and $f^{-1}(B) \in \mathcal{M}$ for all $B \in \mathcal{B}_{\mathbb{R}}$.

Question 3*

The Cantor set may be defined as follows: define $C_0 = [0, 1]$, $C_1 = C_0 \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, ... where C_n is formed from C_{n-1} by removing the open middle third of each of the finitely many closed intervals making up C_{n-1}. This process is illustrated nicely in the following picture from Wikipedia:

The Cantor set is defined by $C = \cap_{n=1}^{+\infty} C_n$.

The Cantor set is a canonical example in because it is "large" in cardinality but "small" both in measure and topologically. In particular, one can show that it is "large" because it has cardinality of the continuum (see Prop 1.22 in Folland). In this question, you will prove the sense in which it is "small."

(a) Prove that C_n is the union of 2^n disjoint closed intervals, each of length $(1/3)^n$. (This follows quickly from the above construction.)

(b) Show that, if $x, y \in C$ and $x < y$, then there exists $z \in (x, y)$, $z \notin C$. In other words, the Cantor set does not contain any intervals. (Since the only connected subsets of C are single points, we say C is *totally disconnected*.)

(c) Show that the Cantor set is closed.

(d) Show that the Cantor set is nowhere dense, that is $\text{int}(\text{cl}(C)) = \emptyset$. (This implies it is "small" topologically.)

(e) Show that $\lambda(C) = 0$. (This implies it is "small" in measure.)

Question 4

Consider Hausdorff topological spaces X and Y. (This ensures that compact sets are closed.) Suppose X is compact and $f : X \to Y$ is continuous and invertible. Prove that f^{-1} is continuous.

Question 5*

The *Cantor function* $f : [0, 1] \to [0, 1]$ may be defined as follows. For each $x \in (\frac{1}{3}, \frac{2}{3})$, let $f(x) = \frac{1}{2}$. Thus f is defined at each point removed from C_0 in the construction of C_1. Next, we define f at each point removed from C_1 in the construction of C_2 by letting $f(x) = \frac{1}{4}$ if $x \in (\frac{1}{9}, \frac{2}{9})$ and $f(x) = \frac{3}{4}$ if $x \in (\frac{7}{9}, \frac{8}{9})$. Continuing in this way, we define $f(x)$ to be $\frac{1}{2^n}$, $\frac{3}{2^n}$, $\frac{5}{2^n}$, ... $\frac{2k-1}{2^n}$, ..., for $k = 1, \ldots, 2^{n-1}$, on the various intervals removed from C_{n-1} in the construction of C_n. This prescribes the values of f on the open set $[0, 1] \setminus C$. By definition, this function is non-decreasing and strictly bounded between 0 and 1 on this set. Finally, we define f on the entire interval $[0, 1]$ by letting $f(0) = 0$ and, for $x \in C \setminus \{0\}$,

$$f(x) = \sup_{t<x} \{f(t) : t \in [0, 1] \setminus C\}.$$

(a) Prove that f is nondecreasing, continuous, and that $f(1) = 1$.

(b) Prove that $g(x) = f(x) + x$ is a bijection from $[0, 1]$ to $[0, 2]$.

(c) Prove that $g^{-1}(x)$ is continuous.

(d) Prove that $\lambda(g(C)) = 1$.

(e) On HW3 Q9, you showed that Lebesgue measure on $\mathcal{M}_{\lambda^*}$ is *complete*, that is, every subset of a Lebesgue null set must be Lebesgue measurable. Use this fact to prove that $g(C)$ contains a set $A \notin \mathcal{M}_{\lambda^*}$ so that $g^{-1}(A) \in \mathcal{M}_{\lambda^*} \setminus \mathcal{B}_{\mathbb{R}}$. *Hint: use HW3 Q5 and Q9.*

(f) Use part (e) to prove that Lebesgue measure on the Borel sets is *not* complete.

The main point of this question is that parts (e-f) demonstrate $\mathcal{B}_{\mathbb{R}} \subsetneq \mathcal{M}_{\lambda^*}$.

Question 6*

Give an example of an *uncountable* collection of Borel functions such that the pointwise supremum is *not* a Borel function.

Hint: consider the indicator function on a nonmeasurable set A, defined by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Question 7

LEMMA 1. *If μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$ and $E \in \mathcal{M}_{\mu^*}$ with $\mu(E) < +\infty$, then for all $\epsilon > 0$, there exists a finite collection of disjoint open intervals $\{I_i\}_{i=1}^n$ so that*

$$\mu\left(E\Delta\left(\cup_{i=1}^n I_i\right)\right) < \epsilon.$$

Prove the lemma.

Question 8*

Consider a measurable space $(X\mathcal{M})$. Given $f, g : X \to \bar{\mathbb{R}}$ measurable, prove the following:

(a) fg is measurable (where $0 \cdot \pm\infty = 0$)

(b) Fix $a \in \bar{\mathbb{R}}$ and define $h(x) = a$ if $f(x) = -g(x) = \pm\infty$ and $h(x) = f(x) + g(x)$ otherwise. Then h is measurable.

Question 9*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

(a) Prove that

$$g(x) = \limsup_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

is Borel measurable. (Hint: Recall HW1, Q8(a) and Midterm 1, Q3(a).)

(b) Let D be the set of points at which f is differentiable. Prove that $D \in \mathcal{B}_{\mathbb{R}}$.