Homework 4 Solutions

3 a

We proceed by induction

Base case: $n=0$, $C_0=[0,1]$ is the union of $2^0=1$ disjoint closed interval of length $(\frac{1}{3})^0=1$.

Inductive step: Assume the result holds for C_{n-1}. By definition C_n is formed from C_{n-1} by removing the open middle third of each interval. Thus the number of intervals doubles and the length of each interval is a third of what it used to be. Also, all of the intervals remain closed and disjoint. Thus, C_n is the union of $2\cdot 2^{n-1}=2^n$ disjoint, closed intervals, each of length $(\frac{1}{3})\cdot(\frac{1}{3})^{n-1}=(\frac{1}{3})^n$.

(b) Suppose $x, y \in C$, $x < y$. Choose $n \in \mathbb{N}$ s.t. $\frac{1}{3^n} < |x-y|$. In the set C_n, there are no intervals longer than $\frac{1}{3^n}$. Thus, x and y belong to disjoint intervals in C_n. Hence $\exists\, z \in (x,y)$ s.t. $z \notin C$.

(c) Each C_n is a finite union of closed intervals, hence is closed. Since C is an intersection of closed sets, it is closed.

(d) By (c), $\text{int}(\text{cl}(C)) = \text{int}(C)$. By part (b), $\text{int}(C) = \emptyset$.

(e) Since C is closed $C \in \mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}_{\lambda^*}$, so C is Lebesgue measurable.

By definition $C_1 \supseteq C_2 \supseteq \ldots$ and $\lambda(C_1) = \lambda([0,1]) = 1$. Thus, by continuity from above,

$$\lambda(C) = \lambda\left(\bigcap_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} \lambda(C_n)$$

$$\overset{(a)}{=} \lim_{n\to\infty} \sum_{i=1}^{2^n} \left(\frac{1}{3}\right)^n = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$

5 a

First, we show f is nondecreasing. The problem states that f is nondecreasing on $[0,1]\setminus C$.

Consequently, if $x \in [0,1]\setminus C$,

$$\sup_{t<x} \{f(t) : t \in [0,1]\setminus C\} \leq f(x). \qquad (*)$$

Since $x \in [0,1]\setminus C = [0,1] \cap \left(\bigcup_{n=1}^{\infty} C_n^c\right)$, $\exists\, n \in \mathbb{N}$ so that $x \in [0,1] \cap C_n^c$. Since this set is open $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq [0,1] \cap C_n^c \subseteq [0,1]\setminus C$. Thus $\{t \in [0,1]\setminus C : t < x\} \cap ([0,1] \cap C_n^c)$ is nonempty and there exists $t \in [0,1]\setminus C$ with $t < x$ s.t. $f(t) = f(x)$.
Hence, equality holds in $(*)$.

Thus, for all $x \in (0,1]$,

$$f(x) = \sup_{t<x} \{f(t) : t \in [0,1]\setminus C\}.$$

Since $x \leq y \Rightarrow$
$\{f(t) : t \in [0,1] \setminus C, t < x\} \subseteq \{f(t) : t \in [0,1] \setminus C, t < y\}$,

this shows f is nondecreasing.

Next, we show f is continuous.

Since f is nondecreasing, $\lim_{y \nearrow x} f(y)$ and $\lim_{y \searrow x} f(y)$ exist, and f is discontinuous at a point x_0 iff $\lim_{y \nearrow x_0} f(y) < \lim_{y \searrow x_0} f(y)$. (If $x_0 = 0$, we use the convention $\lim_{y \nearrow x_0} f(y) = f(0)$ and, similarly, if $x_0 = 1$, we take $\lim_{y \searrow x_0} f(y) = f(1)$.)

Suppose f is discontinuous at $x_0 \in [0,1]$. Then,

$$\left(\lim_{y \nearrow x_0} f(y), \lim_{y \searrow x_0} f(y)\right) \not\subseteq f([0,1])$$

However, every open interval in $[0,1]$ contains a point of the form $\frac{2k-1}{2^n}$ for some $k \in \mathbb{N} \cap [1, 2^{n-1}]$, $n \in \mathbb{N}$. Thus, $\exists$ $k \in \mathbb{N} \cap [1, 2^{n-1}]$ and $x \in [0,1] \setminus C$ s.t.

$$f(x) = \frac{2k-1}{2^n} \in \left(\lim_{y \nearrow x_0} f(y), \lim_{y \searrow x_0} f(y)\right).$$

This is a contradiction. Thus, f is continuous at x_0 for all $x_0 \in [0,1]$.

Finally, the fact that $f(1)=1$ follows from the fact that

$$f(t) < 1 \text{ for } t \in [0,1] \setminus C \Rightarrow f(1) \leq 1$$

$$\exists\, t_n \in [0,1] \setminus C \text{ s.t. } f(t_n) = \frac{2^n - 1}{2^n} \Rightarrow f(1) \geq 1.$$

(b) Since $f(x)$ is nondecreasing, $g(x) = f(x) + x$ is strictly increasing, hence injective.

Since $f(x)$ is continuous, $g(x) = f(x) + x$ is continuous. By the intermediate value theorem, it attains all values in the interval $[g(0), g(1)] = [0,2]$, so it is surjective.

(c) The fact that g is continuous is a consequence of Q4.

(d) First, note that $g(C) = g\left(\bigcap_{n=1}^{\infty} C_n\right) \overset{g \text{ bijective}}{=} \bigcap_{n=1}^{\infty} g(C_n)$

C_n is a union of 2^n closed intervals K_i^n

$$\overset{C_n \text{ is a union of } 2^n \text{ closed intervals } K_i^n}{=} \bigcap_{n=1}^{\infty} g\left(\bigcup_{i=1}^{2^n} K_i^n\right) \underset{g \text{ bijective}}{=} \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{2^n} g(K_i^n)$$

Since g is continuous, $g(K_i^n)$ is compact. Thus, $g(C) \in \mathcal{B}_{\mathbb{R}}$.

Now, by countable additivity,

$$2 = \lambda([0,2]) = \lambda([0,2] \setminus g(C)) + \lambda(g(C))$$

(*) Thus, it suffices to show $\lambda([0,2] \setminus g(C)) = 1$.

To do this, note that $[0,2] \setminus g(C) = g([0,1] \setminus C)$, and recall that $[0,1] \setminus C = \bigcup_{i=1}^{\infty} I_i$, where I_i are disjoint open intervals and f is constant on each interval, i.e. $f(I_i) = c_i$.

Now,

$$\lambda([0,2]\setminus g(C)) = \lambda(g([0,1]\setminus C)) = \lambda\left(g\left(\bigcup_{i=1}^{\infty} I_i\right)\right)$$

$$= \lambda\left(\bigcup_{i=1}^{\infty} g(I_i)\right) \overset{\text{countable additivity}}{=} \sum_{i=1}^{\infty} \lambda(g(I_i)) = \sum_{i=1}^{\infty} \lambda(f(I_i)+I_i)$$

$$= \sum_{i=1}^{\infty} \lambda(c_i + I_i) \overset{\text{translation invariance}}{=} \sum_{i=1}^{\infty} \lambda(I_i) \overset{\text{countable additivity}}{=} \lambda\left(\bigcup_{i=1}^{\infty} I_i\right) = \lambda([0,1]\setminus C)$$

By Q3(e),

$$\lambda([0,2]\setminus g(C)) = \lambda([0,1]\setminus C) = \lambda([0,1]) - \lambda(C) = 1.$$

By HW 3, Q5, since $\lambda(g(C)) > 0$, $\exists$ $A \subseteq g(C)$ s.t. $A \notin \mathcal{M}_{\lambda^*}$.

However $g^{-1}(A) \subseteq C$ and $\lambda(C) = 0$, so we must have $g^{-1}(A) \in \mathcal{M}_{\lambda^*}$, since Lebesgue measure is complete

Note that $g^{-1}(A) \in \mathcal{B}_{\mathbb{R}}$ cannot be true. If it were, then the fact that g^{-1} is continuous, hence Borel measurable, would imply

$$(g^{-1})^{-1}(g^{-1}(A)) = A \in \mathcal{B}_{\mathbb{R}},$$

which contradicts $A \notin \mathcal{M}_{\lambda^*}$.

(f) Since $C \in \mathcal{B}_{\mathbb{R}}$, $\lambda(C) = 0$, $g^{-1}(A) \subseteq C$, and $g^{-1}(A) \notin \mathcal{B}_{\mathbb{R}}$, this shows that not all subsets of Borel null sets are Borel. Thus Lebesgue measure on Borel sets is not complete.

(6) For any $x_0 \in \mathbb{R}$, $1_{\{x_0\}}$ is Borel measurable, since $\{x_0\}$ is closed. However, if $A \notin \mathcal{B}_{\mathbb{R}}$,

$$1_A(x) = \sup_{x_0 \in A} 1_{\{x_0\}}(x)$$

is not Borel measurable, since $1_A^{-1}(\{1\}) = A \notin \mathcal{B}_{\mathbb{R}}$.

(8)

By Q2 (ii), it suffices to show the preimages of $\{-\infty\}$, $\{+\infty\}$, and B are measurable, for all $B \in \mathcal{B}_{\mathbb{R}}$.

(a) $$(fg)^{-1}(\{-\infty\}) = \left(f^{-1}(\{-\infty\}) \cap g^{-1}(\{(0,+\infty]\})\right)$$
$$\cup \left(g^{-1}(\{-\infty\}) \cap f^{-1}(\{(0,+\infty]\})\right)$$
$$\cup \left(f^{-1}(\{+\infty\}) \cap g^{-1}(\{[-\infty,0)\})\right)$$
$$\cup \left(g^{-1}(\{+\infty\}) \cap f^{-1}(\{[-\infty,0)\})\right)$$

which is measurable, since f and g are. Similarly, we see $(fg)^{-1}(\{+\infty\})$ is measurable.

It remains to show $(fg)^{-1}(B) \in \mathcal{M}$, $\forall B \in \mathcal{B}_{\mathbb{R}}$.

Define $\tilde{f}(x) = \begin{cases} f(x) & \text{if } f(x) \in \mathbb{R} \\ 0 & \text{if } f(x) \in \{-\infty, +\infty\} \end{cases} = f(x) 1_{f^{-1}(\{\pm\infty\})}(x)$

and $\tilde{g}(x)$ in the same way.

By Q2(ii), $\tilde{f}^{-1}(\mathcal{B}) = \begin{cases} f^{-1}(\mathcal{B}) & \text{if } 0 \notin \mathcal{B} \\ f^{-1}(\mathcal{B}) \cup f^{-1}(\{\pm\infty\}) & \text{if } 0 \in \mathcal{B} \end{cases}$

is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ measurable. Thus, by the practice midterm, $\tilde{f}\tilde{g}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable.

Since $\tilde{f}(x)\tilde{g}(x) = f(x)g(x)$ if $x \in (fg)^{-1}(\mathbb{R})$, for all $B \in \mathcal{B}_{\mathbb{R}}$,

$$\begin{aligned}(fg)^{-1}(B) &= \{x \in (fg)^{-1}(\mathbb{R}) : f(x)g(x) \in B\} \\ &= \{x \in (fg)^{-1}(\mathbb{R}) : \tilde{f}(x)\tilde{g}(x) \in B\} \\ &= (\tilde{f}\tilde{g})^{-1}(B) \\ &\in \mathcal{M}.\end{aligned}$$

(b)

Recall the notation $\tilde{f}$ and $\tilde{g}$ from the proof of part (a).

Fix $B \in \mathcal{B}_{\mathbb{R}}$. Then

$$(f+g)^{-1}(B) = \begin{cases} (\tilde{f}+\tilde{g})^{-1}(B) & \text{if } a \notin B \\ (\tilde{f}+\tilde{g})^{-1}(B) \cup \{x : f(x) = -g(x) \pm \infty\} & \text{if } a \in B \end{cases}$$

$\in \mathcal{M}$,

by the $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurability of $\tilde{f}$ and $\tilde{g}$, the Practice Midterm, and Q2(ii).

Likewise,

$$(f+g)^{-1}(\{+\infty\}) = \underbrace{\left(f^{-1}(\{+\infty\}) \cap g^{-1}((-\infty,+\infty])\right) \cup \left(g^{-1}(\{+\infty\}) \cap f^{-1}((-\infty,+\infty])\right)}_{\in \mathcal{M}}$$

by Q2 (ii).

Similarly, $(f+g)^{-1}(\{-\infty\}) \in \mathcal{M}$.

Thus, a final application of Q2 (ii) ensures $(f+g)$ is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$ measurable.

(10) By definition,

$$g(x) = \inf_{h>0} \sup_{0<\delta<h} \frac{f(x+\delta)-f(x)}{\delta}$$

Since $h \mapsto \sup_{0<\delta<h} \frac{f(x+\delta)-f(x)}{\delta}$ is decreasing as $h \to 0$, its limit as $h \to 0$ exists and

$$g(x) = \lim_{n\to\infty} \sup_{0<\delta<\frac{1}{n}} \frac{f(x+\delta)-f(x)}{\delta}$$

Since f is continuous, $\forall \delta > 0$

$$x \mapsto \frac{f(x+\delta)-f(x)}{\delta}$$

is continuous. Thus, $\forall n \in \mathbb{N}$, HW1 Q8 ensures

$$x \mapsto \sup_{0<\delta<\frac{1}{n}} \frac{f(x+\delta)-f(x)}{\delta}$$

is lower semicontinuous. By Mid 1, Q3(a), it is Borel measurable. Thus, $g(x)$ is the limit of measurable functions, hence measurable.

(b) Since a function $h: \mathbb{R} \to \overline{\mathbb{R}}$ is Borel meas iff $-h$ is Borel meas, applying the result of part (a) to $-f$, we see that

$$\liminf_{h\to 0} \frac{f(x+h)-f(x)}{h} = -\limsup_{h\to 0} \frac{-f(x+h)+f(x)}{h}$$

is Borel meas. By definition,

$$D^c = \{x : \underbrace{\liminf_{h\to 0} \frac{f(x+h)-f(x)}{h}}_{g_1(x)} \neq \underbrace{\limsup_{h\to 0} \frac{f(x+h)-f(x)}{h}}_{g_2(x)}\},$$

so D^c is measurable, hence D is measurable.