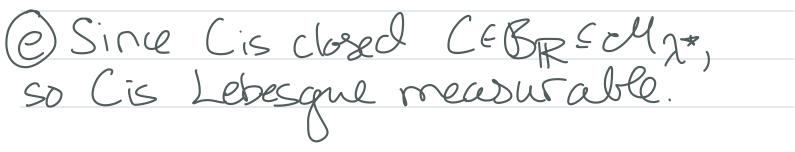
Homework 4 Solutions © Katy Craig, 2024 (3)a)We proceed by induction Base case: n=0, (o=[0,1]) is the union of $2^{\circ}=1$ disjoint closed interval of length($5^{\circ}=1$. Inductive step: Assume the result holds for Cn-1. By definition Cn is formed from Cn-1 by removing the open middle third of each interval. Thus the number of intervals desubles and the length of each interval is a third of what it used to be. Also, all of the intervals remain closed and disjoint Cn is the union of $2 \cdot 2^{n-1} = 2^n$ disjoint, desed intervals, each of length $(\frac{1}{3}) \cdot (\frac{1}{3})^{n-1} = (\frac{1}{3})^n$

(b) Suppose x, yEC, x<y. Choose nEINS.t. 3n < hr-yl. In the set (n, there are no intervals longer than 3n. Thus, x and y belong to disjuint intervals in Cn. Hence $\exists 0 \not\equiv f(x,y) \quad s.t. \quad \not\equiv \not\in C$.

(2) Each Cn is a finite union of cloyed intervals, hence is closed. Since C is an intersection of closed sets, it is cloyed. (d) $B_{4}(C)$, int(cl(c)) = int(c). By part (b), $int(c) = \beta$.



By definition $C_1 \ge C_2 \ge ...$ and $\lambda(C_1) = \lambda(E_0, []) = 1$. Thus, by condinuity from above, $\lambda(C) = \lambda(\bigwedge_{n=1}^{\infty} C_n) = \lim_{n \to \infty} \lambda(C_n)$ (a) 2^{-1} = $\lim_{n \to \infty} \sum_{j=1}^{n} (\frac{1}{2})^n = \lim_{n \to \infty} (\frac{2}{3})^n = 0.$

(5)a)First, we show f is nonebereasing. The problem states that f is nondecreasing on [0,1]\C. Consequently, if $x \in [0,1] \setminus C_j$ sup $9f(t): t \in [0,1] \setminus C_j^2 \leq f(x)$. t < x(*)

Since $x \in [0,1] \setminus C = [0,1] \cap (\bigcup_{n=1}^{U} Cn^{c}), \exists n \in \mathbb{N}$ so that XE[0,1] (Cn. Since this set is open $\exists \epsilon > 0$ s.t. $B_{\epsilon}(x) \in [0,1] \cap (n^{c} \in [0,1]) C.$ Thus {te[0,1] (C:t<x} ([0,1] (n)) is nonempty and there exists te [0,] (C with $t < x \quad s.t. \quad f(t) = f(x)$. Hence, equality holds in (+). Thus, for all x ∈ (0, 1], $f(x) = \sup_{t < x} \{f(t) : t \in [0, 1] \setminus C\}.$

Since $\chi \leq q =$ $\{f_{t}\}: \{e[0,1] \mid C, t < \chi\} \leq \{f_{t}\}: \{e[0,1] \mid C, t < q\},$

this shows fis nondecreasing.

Next, we show f is continuous. Since f is nondecreasing, $y^{2}x f(y)$ and $y^{3}x f(y)$ exist, and f is discontinuous at a point x_{0} iff $y^{2}x_{0}f(y) \leq y^{3}x_{0}f(y)$. (If $x_{0}=0$, we use the convention $y^{2}x_{0}f(y)=f(0)$ and, similarly, If $x_{0}=1$, we take $y^{3}x_{0}f(y)=f(1)$.)

Suppose fis discontinuous at xo ELO, 1]. Then,

(lim yarofly), yarofly) & f([0,1])

However, every open interval in [0,1]contains a point of the form 2^{n} for some $k \in [N \cap [1, 2^{n-1}], n \in [N]$. Thus, $\exists k \in [N \cap [1, 2^{n-1}]$ and $x \in [0, \overline{D} \setminus C$ s.t.

 $f(\chi) = \frac{2k-1}{2^m} \in \left(\lim_{y \to \infty} f(y), \lim_{y \to \infty} f(y) \right).$

This is a contradiction. Thus, f is continuous at xo for all xo EG, J.

Finally, the fact that f(1)=1 follows) from the fact that

f(t) < 1 for $t \in [0, 1] \setminus C = > f(1) \leq 1$

 $\exists t_n \in [0, J] (c_{s,t}, f(t_n) = \frac{2^{n-1}}{2^n} =) f(i) \ge 1.$

Dince fbc) is nondecreasing, g(x)=f(x)+x is strictly increasing, hence injective.

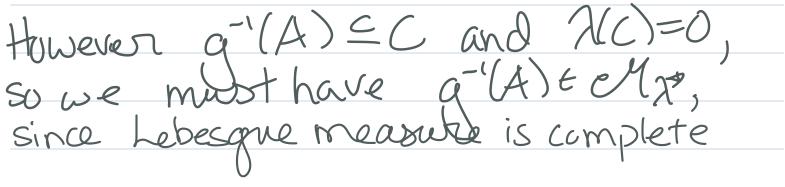
Since f(x) is continuous, g(x) = f(x) + x is continuous. By the intermediate value theorem, it attains all values in the interval Lg(o), g(1) = [0, 2], so it is surjective.

O the fact that q is continuous is a consequence of 024.

8 bijective Since q is continuous, $q(K_i^m)$ is compact. Thus, $q(c) \in B_{\mathbb{R}}$. Now, by countable additivity, $2 = \lambda([0,2]) = \lambda([0,2])q(c)) + \lambda(q(c))$ (A) Thus, it suffices to show $\lambda([0,2] \setminus q(C)) = 1$. To do this, note that $[0,2]\setminus q(c)=q(0,1)(c)$, and recall that $[0,1]\setminus c=[0,1]$, where Ti are disjoint open intervals and f is constant on each interval, i.e. f(I:)=Ci.

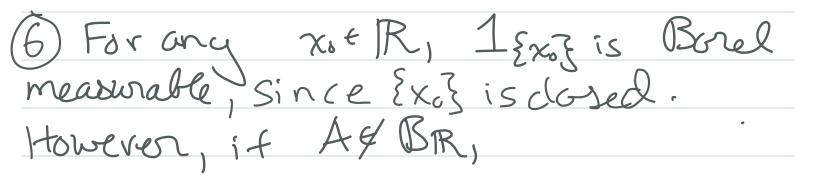
Now,

$$\begin{split} \lambda([o_12] \setminus q(c)) &= \lambda(q([o_1D] \setminus c)) = \lambda(q([i_{i=1}^{\infty}]I_i)) \\ &= \lambda([i_{i=1}^{\infty}q([i_{i}])] = \sum_{i=1}^{\infty} \lambda(q([i_{i}])) = \sum_{i=1}^{\infty} \lambda(f([i_{i}]) + [i_{i}]) \\ &= \sum_{i=1}^{\infty} \lambda(c_{i} + [i_{i}]) = \sum_{i=1}^{\infty} \lambda([i_{i}]) = \lambda([i_{i}] + [i_{i}]) \\ &= \sum_{i=1}^{\infty} \lambda(c_{i} + [i_{i}]) = \sum_{i=1}^{\infty} \lambda([i_{i}]) = \lambda([i_{i}] + [i_{i}]) \\ &= \sum_{i=1}^{\infty} \lambda(c_{i} + [i_{i}]) = \sum_{i=1}^{\infty} \lambda([i_{i}]) = \lambda([i_{i}]) \\ &= \lambda([i_{i}]) = \lambda([i_{i}]) \\ &= \lambda([i_{i}]) = \lambda([i_{i}]) \\ &= \lambda([i_{i}]) \\ &= \lambda([i_{i}]) = \lambda([i_{i}]) \\ &= \lambda([i_{i}])$$
By Q3@, $\lambda([0,2])_{q(c)}=\lambda([0,1])_{c}=\lambda([0,1])_{c}=1.$ By HW3, QS, since $\lambda(q(c))>0$, JAS q(c) s.t. AE M2.



Note that q (A) & BR cannot be true. If it were, then the fact that g-1 is continuous, here Borel measurable, would imply (g⁻)⁻ (g⁻(A)) = A E B_R, which contradicts A & M₂^{*}.

(f) Since CEBR, A(c)=0, q=(A)=C, and q'(A) & BR, this shows that not all subsets of Borel null sets are Borel. Thus Lebesque measure on Borel sets is not complete.



 $I_A(x) = \sup I_{\xi x \xi}(x)$ xot A

is not Korel measurable, since 1_A({{1}}) = A & BR.

8 By Q2 (ii), it suffices to show the preimages of 2-003, Etas, and B are measurable, for all BEBR.

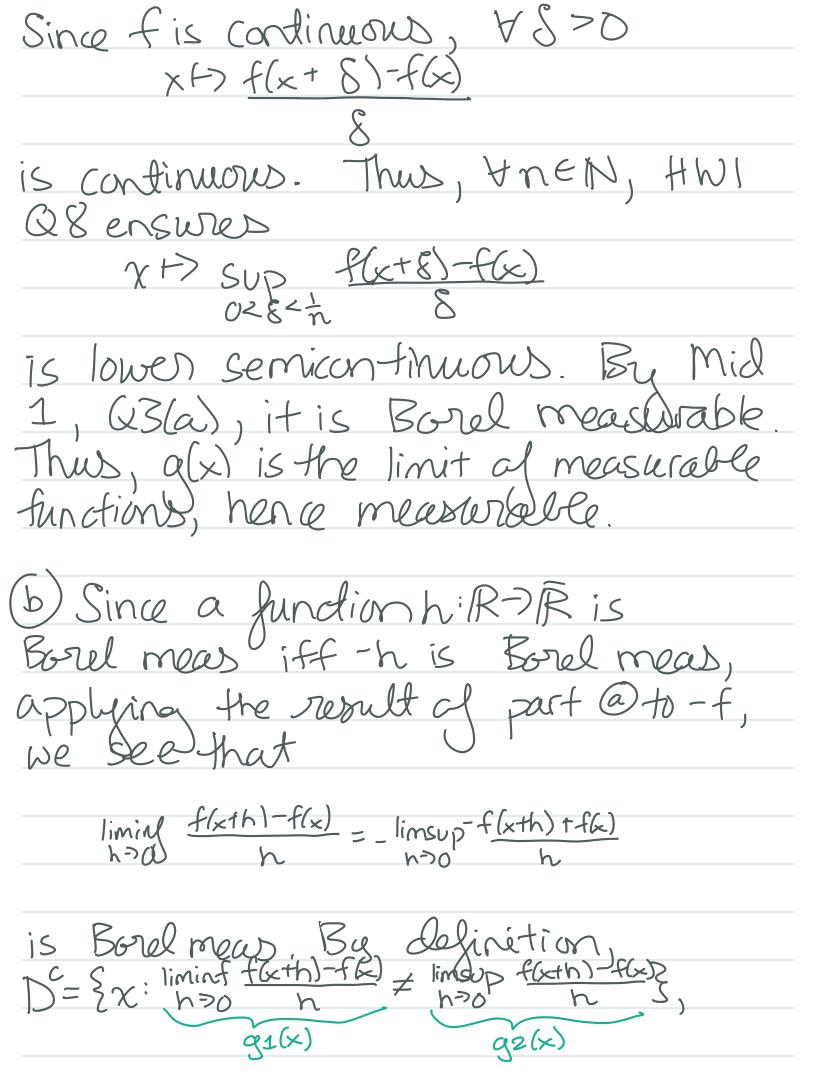
 $(a) (f_{a})^{-1}(\xi - \omega \xi) = (f^{-1}(\xi - \omega \xi)) \cap a^{-1}(\xi(0, + \omega \xi))$ $\cup (q^{-1}(\xi - \omega \xi)) \cap f^{-1}(\xi(0, + \omega \xi))$ $\cup (q^{-1}(\xi - \omega \xi)) \cap f^{-1}(\xi(0, + \omega \xi))$ $U(f)^{-1}(\xi + \alpha \xi) \cap q^{-1}(\xi (-\alpha \delta, 0)))$ $U(q^{-1}(\xi + \alpha \overline{\zeta}) \wedge (f^{-1}(\xi - \alpha \overline{\zeta})))$ which is measurable, since f and gare. Similarly, we see (fg) (t 2) Is measurable.

It remains to show (fg) (B) EM, V RERD VBEBR.

Define $f(x) = \{f(x) \mid i \neq f(x) \in \mathbb{R} = f(x) \mid f(x) \neq 0\}$ $\begin{cases} 0 \quad i \neq f(x) \neq \xi = \infty, \pm \infty \\ 0 \quad i \neq f(x) \neq \xi = \infty, \pm \infty \end{cases}$ and g(x) in the same way.

By Q2(iii), $\tilde{f}^{-1}(B) = \int f^{-1}(B) if O \notin B$ $(f^{-1}(B) \cup f^{-1}(E \pm \infty)) if O \in B$ is (M, BR) measurable. Thus, by the practice midtern, Fg is (M, BR)-measurable. Recall the notation I and g from the proof of part @. Fix BEBR Then $(f+q)^{-1}(B) = S(f+q)^{-1}(B)$ if $a \notin B$ $((f+q)^{-1}(B)U \le \chi; f(\chi) = -g(\chi) \pm -\infty)^{-1} if a \notin B$ -11EM, by the (M, BR)-measurability of Fland q, the Practice Midterm, and Q2(ii).

hikewise, $[f+g]^{-1}(\xi+\omega] = (f^{-1}(\xi+\omega]) \cap g^{-1}([-\omega],+\omega])$ $\cup (g^{-1}(\xi+\omega) \cap f^{-1}([-\omega],+\infty]))$ $\in \mathcal{M}$ by Q2(ii). Similarly, $(f+q)^{-1}(\xi-\sigma\xi) \in \mathcal{M}$. thus, a final application of G2(ii) ensures (frg) is (ell, Bird measurable. (10) By definition, $g(x) = \inf_{h>0} \sup_{0 \le ch} f(x+8) - f(x)$ $h>0 = h>0 = 0 \le ch$ Since $h \to 0 \le ch$ $s = \int_{0}^{1} \frac{f(x+8) - f(x)}{8} = \int_{$ g(x)= lim SUP <u>f(x+8)-f(x)</u> n>00 028



so D'is measurable, hence Dis measurable.