MATH 201A: HOMEWORK 5

Due Sunday, November 10th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice questions will appear on each exam. *All answers should be justified with either a proof or a counterexample.*

Question 1*

Consider a measure space $(X, \mathcal{M}, \mu)$. Suppose $f : X \to \overline{\mathbb{R}}$ is measurable.

(a) If $f \geq 0$ and $\int f d\mu < +\infty$, prove that $\{x : f(x) = +\infty\}$ is a null set.

(b) Under the same hypotheses of part (a), prove that $\{x : f(x) > 0\}$ is σ-finite.

(c) If $f \in L^1(\mu)$, prove that there exists $g : X \to \mathbb{R}$ so that $g = f$ μ-a.e..

Question 2*

Let μ be a locally finite measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. (See HW4, Q1 for the definition and some key facts.) Define

$$B(x,r) = \{y : |y-x| < r\}, \quad \bar{B}(x,r) = \{y : |y-x| \leq r\}.$$

(a) Prove that, for all $r > 0$, $x \mapsto \mu(B(x,r))$ is lower semicontinuous.
Hint: First, prove that $\mu(B(x,r)) = \lim_{n\to+\infty} \mu(B(x, r-1/n))$.

(b) Prove that, for all $r > 0$, $x \mapsto \mu(\bar{B}(x,r))$ is upper semicontinuous.
Hint: How can you adapt the hint from part (a)?

(c) For any $k \geq 0$, prove that

$$f(x) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^k}$$

is Borel measurable.
Hint: Write $f(x) = \lim_{r\to 0} G_r(x)$, *where*

$$G_r(x) = \sup\left\{\frac{\mu(B(x,\rho))}{\rho^k} : 0 < \rho < r\right\}.$$

Question 3

(a) Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, $\psi : \mathbb{R} \to \mathbb{R}$, and $\phi = \psi$ Lebesgue almost everywhere. Then ψ is Lebesgue measurable.

(b) Now, suppose $\phi : \mathbb{R} \to \mathbb{R}$ is Borel measurable, $\psi : \mathbb{R} \to \mathbb{R}$, and $\phi = \psi$ Borel almost everywhere. Is ψ is Borel measurable? Justify your answer with a proof or counterexample.

Question 4*

(a) Given a Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ and a Lebesgue measurable set E prove that

$$\int_E f(x) d\lambda(x) = \int_{E+c} f(x-c) d\lambda(x) \quad \text{for all } c \in \mathbb{R},$$

whenever the integral $\int_E f(x) d\lambda(x)$ exists.

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function that is differentiable Lebesgue almost everywhere. (In fact, you can prove in a later measure theory course that all nondecreasing functions are differentiable Lebesgue-almost everywhere!) Prove that

$$\int_{[0,1]} \limsup_{n \to +\infty} \frac{f(x+1/n) - f(x)}{1/n} \, d\lambda(x) \leq f(1) - f(0)$$

Hint: It can be convenient to assume, without loss of generality, that $f(x) = f(1)$ for all $x \geq 1$. Use part (a).

Question 5

Consider a measure space $(X, \mathcal{M}, \mu)$ and a sequence of nonnegative, measurable functions f_n satifying

$$f_n(x) \searrow f(x) \text{ for all } x \in X \quad \text{and} \quad \int f_1 < +\infty.$$

Prove that

$$\lim_{n \to +\infty} \int f_n = \int \lim_{n \to +\infty} f_n.$$

Question 6*

Consider a measure space $(X, \mathcal{M}, \mu)$ and a nonnegative, measurable function $f \in L^1(\mu)$. Prove that, for all $\epsilon > 0$, there exists $A \in \mathcal{M}$ so that $\mu(A) < +\infty$ and $\int_A f > \int f - \epsilon$.

Hint: use the definition of the integral in terms of simple functions, and choose a simple function with integral close to $\int f d\mu$.

Question 7*

Prove the following *generalized dominated convergence theorem*:

THEOREM 1. *Consider a measure space $(X, \mathcal{M}, \mu)$. Suppose $f_n, g_n, f, g \in L^1(\mu)$ are functions for which the following hold:*

(i) $f_n \to f$ *μ-a.e.* ,

(ii) $g_n \to g$ *μ-a.e.* ,

(iii) $|f_n| \leq g_n$ *μ-a.e.* ,

(iv) $\int g_n d\mu \to \int g d\mu$.

Then $\int f_n d\mu \to \int f d\mu$.

Question 8*

Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f : X \to [0, +\infty]$ satisfies $f \in L^1(\mu)$. Prove that, for all $\epsilon > 0$, there exists $\delta > 0$ so that if $E \in \mathcal{M}$ satisfies $\mu(E) < \delta$, then $\int_E f d\mu < \epsilon$.

Hint: Proceed by contradiction, so that you can find a sequence $E_n \in \mathcal{M}$ with arbitrarily small measure so that $\int_{E_n} f d\mu \geq \epsilon$. Consider the set $A = \cap_{n=1}^{+\infty} \cup_{k>n} E_k$.

Question 9

Consider a measure space $(X, \mathcal{M}, \mu)$.

(a) Fix a nonnegative, measurable function f. Prove that $\nu(A) := \int_A f d\mu$ is a measure on $\mathcal{M}$.

(b) Suppose $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \lambda)$, where λ is Lebesgue measure. Fix $f \in L^1(\lambda)$. Consider the cumulative distribution function

$$F(x) := \int_{(-\infty, x]} f d\lambda.$$

Prove that F is continuous.

(c) Given an example of a Borel measure μ on the real line for which the cumulative distribution function

$$F(x) := \int_{(-\infty, x]} d\mu.$$

is *not* continuous.

Question 10

In this question, you will show why the $\liminf$ in Fatou's lemma can't be naively replaced with a $\limsup$. ~~Fix~~ Give an example of a measure space $(X, \mathcal{M}, \mu)$ and sequence of nonnegative measurable functions $\{f_n\}_{n \in \mathbb{N}}$ so that ~~f_n converges pointwise to some f but~~

$$\limsup_{n \to +\infty} \int f_n d\mu < \int \limsup_{n \to +\infty} f_n d\mu.$$