MATH 201A: HOMEWORK 6

Due Sunday, November 17th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice questions will appear on each exam. *All answers should be justified with either a proof or a counterexample.*

Question 1*

Consider a measure space $(X, \mathcal{M}, \mu)$ with $\mu(X) < +\infty$.

In this question, you will prove that convergence in measure is *metrizable*: there exists a metric on the space of measurable functions (up to almost everywhere equivalence) so that convergence in this metric is equivalent to convergence in measure.

(a) Define $\phi : [0, +\infty) \to [0, +\infty)$ by $\phi(s) = s/(1+s)$. Prove that $\phi(s)$ is nondecreasing, $\phi(s+t) \leq \phi(s) + \phi(t)$ and $\phi(s) = 0 \iff s = 0$.

(b) Given $f, g : X \to \mathbb{R}$ measurable, define

$$\rho(f, g) = \int \frac{|f-g|}{1+|f-g|} d\mu.$$

Prove that ρ is a metric on the space of measurable functions, if we identify functions that are equal almost everywhere.

(c) Given $f_n, f : X \to \mathbb{R}$ measurable, show that

$$\rho(f_n, f) \to 0 \iff f_n \to f \text{ in measure.}$$

Question 2*

Recall the definition of the Riemann integral from HW1, Q3. Suppose that f is a bounded, real-valued function on $[a, b] \subseteq \mathbb{R}$. If it is Riemann integrable, we will denote its Riemann integral by $\int_a^b f(x)dx$.

(a) Prove that, if f is Riemann integrable on $[a, b]$, then f is Lebesgue integrable on $[a, b]$, and

$$\int_a^b f(x)dx = \int_{[a,b]} f(x) d\lambda(x)$$

Hint:

(i) Define simple functions $u_P(x)$ and $l_P(x)$ so that $U(P, f) = \int u_P(x) d\lambda(x)$ and $L(P, f) = \int l_P(x) d\lambda(x)$.

(ii) Choose a sequence of partitions P_k that attains the infimum and the supremum in the definition of the upper and lower Riemann integrals, where P_{k+1} is a refinement of P_k for all $k \in \mathbb{N}$. Argue that this ensures $u_{P_k}(x)$ is decreasing while $l_{P_k}(x)$ is increasing and define their respective limits to be $u(x)$ and $l(x)$

(iii) Explain why $l(x) \leq f(x) \leq u(x)$ for $x \in [a,b]$ and $\int u(x)d\lambda(x) = \int l(x)d\lambda(x) = \int_a^b f(x)dx$. Show this implies that that $u = f$ almost everywhere in $[a,b]$.

(iv) Use Q1 to conclude that f is Lebesgue measurable. Use this to complete the proof.

(b) Consider the function $f(x) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n} 1_{(n,n+1]}(x)$. Since f is piecewise continuous, it is Riemann integrable on $[0,b]$ for all $b > 0$. Prove that $\lim_{b\to+\infty} \int_0^b f(x)dx$ exists, but $f \notin L^1(\lambda)$.

Question 3*

In this problem, you may use the following fact without proof:

$$||a+b| - |a| - |b|| \leq 2|b| \ , \quad \forall a, b \in \mathbb{R}.$$

Consider a measure space $(X, \mathcal{M}, \mu)$ and $\{f_n\}_{n=1}^\infty \subseteq L^1(\mu)$ satisfying $f_n \to f$ μ-almost everywhere.

(a) Suppose there exists $M > 0$ so that $\|f_n\|_{L^1(\mu)} \leq M$ $\forall n \in \mathbb{N}$. Prove that

$$f \in L^1(\mu) \quad \text{and} \quad \lim_{n\to+\infty} \int (|f_n| - |f_n - f|) \, d\mu = \int |f| d\mu.$$

Hint: take $a = f_n - f$ and $b = f$.

(b) Suppose that $f \in L^1(\mu)$ and $\lim_{n\to+\infty} \|f_n\|_{L^1(\mu)} = \|f\|_{L^1(\mu)}$. Prove that

$$\lim_{n\to+\infty} \|f_n - f\|_{L^1(\mu)} = 0.$$

Question 4

Suppose $\{f_n\}_{n=1}^\infty \subseteq L^1(\mu)$ and $f_n \to f$ uniformly.

(a) If $\mu(X) < +\infty$, prove that $f \in L^1(\mu)$ and $\int f_n d\mu \to \int f d\mu$.

(b) If $\mu(X) = +\infty$, give a counterexample to show the result of part (a) fails.

Question 5

Suppose μ is the counting measure on $\mathbb{N}$ endowed with the σ-algebra $2^{\mathbb{N}}$. Prove that $f_n \to f$ in measure if and only if $f_n \to f$ uniformly

Question 6

Suppose $f_n \geq 0$ is a sequence of measurable functions satisfying $f_n \to f$ in measure. Prove that $\int f \leq \liminf_{n\to+\infty} \int f_n$.

Question 7

Consider a sequence of measurable sets E_n and suppose that $\mu(E_n) < +\infty$ for all $n \in \mathbb{N}$ and $1_{E_n} \to f$ in $L^1(\mu)$. Prove that there exists a measurable set E so that $f = 1_E$ μ-a.e..

Question 8*

Consider the sequence of functions

$$f_n(x) = \frac{\sin(x/n)}{x/n}(1+x)^{-2}1_{[0,+\infty)}(x).$$

We consider the expression $\sin(x)/x$ as an abbreviation for the function that takes the value $\sin(x)/x$ for $x \neq 0$ and 1 for $x = 0$.

(a) Are the functions f_n Borel measurable? Are they Lebesgue measurable? Justify your answers.

(b) Compute the following limit and justify your calculations. Explain where you use Q2 and standard Calculus facts about the Riemann integral.

$$\lim_{n\to+\infty} \int_{[0,+\infty)} n\sin(x/n)\left[x(1+x)^2\right]^{-1} d\lambda(x)$$

Question 9*

Consider a measure space $(X, \mathcal{M}, \mu)$. Suppose $f_n \to f$ in measure and $g_n \to g$ in measure.

(a) Prove that $f_n + g_n \to f + g$ in measure.

(b) Prove that $f_n g_n \to fg$ in measure if $\mu(X) < +\infty$.

(c) Give a counterexample to show that the result of part (b) does not hold if $\mu(X) = +\infty$.

Question 10

Compute the following limit and justify your calculations. You do not need to explain why the functions are measurable. Explain where you use the equivalence of the Riemann and Lebesgue integral, as well as where you use standard Calculus facts about the Riemann integral.

$$\lim_{n\to+\infty} \int_0^\infty (1+(x/n))^{-n}\sin(x/n)d\lambda(x).$$