Homework 6 Solutions

①

ⓐ $\phi'(s) = \frac{1+s-s}{(1+s)^2} = \frac{1}{(1+s)^2} \geq 0 \Rightarrow \phi$ is nondecreasing

$\phi(s+t) = \frac{s+t}{1+s+t} = \frac{s}{1+s+t} + \frac{t}{1+s+t} \leq \frac{s}{1+s} + \frac{t}{1+t} = \phi(s)+\phi(t)$

$\phi(s) = 0 \Leftrightarrow \frac{s}{1+s} = 0 \Leftrightarrow s=0$, since $s \geq 0 \Leftrightarrow 1+s>0$.

ⓑ $p(f,g)=0$

$\Leftrightarrow \underbrace{\frac{|f-g|}{1+|f-g|}}_{\psi :=} = 0$ a.e., since ψ is a nonneg, meas fn.

$\Leftrightarrow \phi(|f(x)-g(x)|) = 0$ a.e.

$\Leftrightarrow |f(x)-g(x)| = 0$ a.e.

$\Leftrightarrow f=g$ a.e.

$p(f,g) = \int \phi(|f-g|)d\mu = \int \phi(|f-h+h-g|)d\mu$

ϕ nondecreasing

$\leq \int \phi(|f-h|+|h-g|)d\mu$

$$\overset{(a)}{\leq} \int \phi(|f-h|) + \phi(|h-g|)\,d\mu$$

$$\overset{\text{Beppo Levi}}{\leq} \int \phi(|f-h|)\,d\mu + \int \phi(|h-g|)\,d\mu$$

$$= \rho(f,h) + \rho(h,g)$$

(c) Note that

$$\int \phi(|f_n-f|)\,d\mu \geq \int_{\{x:|f_n(x)-f(x)|\geq\varepsilon\}} \phi(|f_n-f|)\,d\mu \overset{\phi \text{ nondecreasing}}{\geq} \int_{\{x:|f_n(x)-f(x)|\geq\varepsilon\}} \phi(\varepsilon)\,d\mu$$

$$\geq \frac{\varepsilon}{1+\varepsilon}\,\mu(\{x:|f_n(x)-f(x)|\geq\varepsilon\}).$$

Thus, $\rho(f_n,f)\to 0$ implies $f_n\to f$ in measure.

Conversely, for all $\varepsilon>0$.

$$\int \phi(|f_n-f|)\,d\mu \leq \int_{\{x:|f_n(x)-f(x)|\geq\varepsilon\}} \phi(|f_n-f|)\,d\mu$$

$$+ \int_{\{x:|f_n(x)-f(x)|<\varepsilon\}} \phi(|f_n-f|)\,d\mu$$

$\phi\leq 1$

ϕ increasing

$$\leq \int_{\{x:|f_n(x)-f(x)|\geq\varepsilon\}} 1\,d\mu$$

$$+ \int_{\{x:|f_n(x)-f(x)|<\varepsilon\}} \phi(\varepsilon)\,d\mu$$

$$\leq \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) + \frac{\varepsilon}{1-\varepsilon}\mu(X)$$

Thus,

$$\limsup_{n\to\infty} \int \phi(|f_n - f|)\,d\mu \leq \frac{\varepsilon}{1-\varepsilon}\mu(X).$$

Since $\varepsilon > 0$ was arbitrary and $\mu(X) < +\infty$, this gives the result.

(2)

(a) By definition,

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i \quad , \quad M_i = \sup_{x_{i-1} \le x \le x_i} f(x)$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i \quad , \quad m_i = \inf_{x_{i-1} \le x \le x_i} f(x).$$

Thus, if we define

$$u_P(x) = \sum_{i=1}^{n} M_i \mathbb{1}_{[x_{i-1}, x_i]} \, , \quad l_P(x) = \sum_{i=1}^{n} m_i \mathbb{1}_{[x_{i-1}, x_i]} \, ,$$

we have $U(P,f) = \int u_P \, d\lambda$, $L(P,f) = \int l_P \, d\lambda$.

Since f is Riemann integrable,

$$(\star) \quad \int_a^b f(x)\,dx = \inf_P U(P,f) = \sup_P L(P,f).$$

Recall that one partition P is a *refinement* of another partition $\tilde{P}$ if $\{x_0, x_1, \ldots, x_n\} \supseteq \{\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{\tilde{n}}\}$.

Note that this implies

(**) $\mathcal{U}(\tilde{P}, f) \geq \mathcal{U}(P, f)$
$L(\tilde{P}, f) \leq L(P, f)$

Choose sequences of partitions P_k^{u}, P_k^{l} that attain the infimum and supremum in (*),

$$\lim_{k\to\infty} \mathcal{U}(P_k^u, f) = \inf_P \mathcal{U}(P, f), \quad \lim_{k\to\infty} L(P_k^l, f) = \sup_P L(P, f).$$

By (**), refining a partition only makes the upper sum smaller and the lower sum bigger. Thus, we may define a new sequence P_k s.t. P_k is a refinement of P_k^u and P_k^l for all $k \in \mathbb{N}$,
and

$$\lim_{k\to\infty} \mathcal{U}(P_k, f) = \inf_P \mathcal{U}(P, f) = \sup_P L(P, f) = \lim_{k\to\infty} L(P_k, f).$$

Similarly, we may assume WLOG that P_{k+1} is a refinement of P_k for all $k \in \mathbb{N}$. This last fact ensures that $u_{P_k}(x)$ and $l_{P_k}(x)$ are decreasing/increasing, by definition of M_i and m_i. Since f is bounded, for each x, $\lim_{k\to\infty} u_{P_k}(x) =: u(x)$ and $\lim_{k\to\infty} l_{P_k}(x) =: l(x)$ exist and are real numbers.

Since $l_{P_k}(x) \le f(x) \le u_{P_k}(x)$ $\forall x \in [a,b]$ we have

(***) $l(x) \le f(x) \le u(x)$ $\forall x \in [a,b]$.

Furthermore,

$$\int u \, d\lambda \underset{\text{Dominated convergence theorem, since } |u_k| \le 1_{[a,b]} \|f\|_\infty \in L^1(\mu)}{=} \lim_{k\to\infty} \int u_k \, d\lambda = \lim_{k\to\infty} U(P_k, f) = \int_a^b f(x)\,dx$$

Dominated convergence theorem, since $|u_k| \le 1_{[a,b]} \|f\|_\infty \in L^1(\mu)$

$$\int l \, d\lambda = \lim_{k\to\infty} \int l_k \, d\lambda = \lim_{k\to\infty} L(P_k, f) = \int_a^b f(x)\,dx$$

Dominated convergence theorem, where l_k has the same dominating fn as u_k above

In particular, we obtain $l \le u$, $\int l \, d\lambda = \int u \, d\lambda$, so $u - l \ge 0$ and $\int u - l \, d\lambda = 0$.

Thus, $u = l$ λ-a.e., so by $(***)$, $u = f$ a.e. in $[a,b]$, and f is measurable.

Finally, by $(***)$ and the fact $\int l \, d\lambda = \int u \, d\lambda$, we conclude,

$$\int_{[a,b]} f \, d\lambda = \int f \, 1_{[a,b]} \, d\lambda = \int u \, d\lambda = \int_a^b f(x) \, dx.$$

Lastly, since f is bounded and measurable, we have $\int_{[a,b]} |f| < +\infty$, so f is integrable on $[a,b]$.

b) By definition,

$$\int_0^b f(x) \, dx = \sum_{n=1}^{\lfloor b-1 \rfloor} \frac{(-1)^n}{n} + \frac{(-1)^{\lfloor b \rfloor}}{\lfloor b \rfloor} (b - \lfloor b \rfloor)$$

Thus, $\lim_{b \to \infty} \int_0^b f(x) \, dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} < +\infty$ by convergence of alternating series. On the other hand

$$\int |f| \, d\lambda \overset{\text{Beppo-Levi}}{=} \sum_{n=1}^{\infty} \int \frac{1}{n} 1_{(n,n+1]} \, d\lambda = \sum_{i=1}^{\infty} \frac{1}{n} = +\infty,$$

so $f \notin L^1(\lambda)$.

(3)

(a) Since $f_n \to f$ μ-a.e. there exists $N \in \mathcal{M}$ with $\mu(N)=0$ so that $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in N^c$. Then $1_{N^c} f_n$ is measurable and $\lim_{n\to\infty} 1_{N^c} f_n(x) = 1_{N^c} f(x)$ $\forall x \in X$, so $1_{N^c} f(x)$ is measurable and $f = 1_{N^c} f$ a.e.

Furthermore, by Fatou,

$$\int |f| 1_{N^c} \, d\mu \le \liminf_{n\to\infty} \int |f_n| 1_{N^c} \, d\mu \le M,$$

so $f 1_{N^c} \in L^1(\mu)$, which implies $f \in L^1(\mu)$.

Using the inequality at the top of the problem and the Dominated Convergence Thm,

(*)

$$\lim_{n\to\infty} \int \big| |f_n| - |f_n - f| - |f| \big| = \int \lim_{n\to\infty} \big| |f_n| - |f_n - f| - |f| \big| = 0$$

$$\parallel$$

$$\limsup_{n\to\infty} \int \big| |f_n| - |f_n - f| - |f| \big|$$

$$\text{VI}$$

$$\limsup_{n\to\infty} \int |f_n| - |f_n - f| - |f| = \limsup \int |f_n| - |f_n - f| - \int |f|$$

Thus, $\limsup \int |f_n| - |f_n - f| \leq \int |f|$

On the other hand (*) ensures

$$\lim_{n\to\infty} -\int \big| |f_n| - |f_n - f| - |f| \big| = 0$$
$$\shortparallel$$
$$\liminf -\int \big| |f_n| - |f_n - f| - |f| \big|$$
$$\wedge |$$
$$\liminf \int |f_n| - |f_n - f| - |f|$$
$$\shortparallel$$
$$\liminf \int |f_n| - |f_n - f| - \int |f|$$

Thus, $\liminf \int |f_n| - |f_n - f| \geq |f|$

(b) Since $f \in L^1(\mu)$ and $\|f_n\|_{L^1(\mu)} \to \|f\|_{L^1(\mu)}$, $\exists\ M > 0$ s.t. $\|f_n\|_{L^1(\mu)} \leq M\ \forall n \in \mathbb{N}$.

Thus, we may apply part (a). Thus,

$$\int |f| = \lim_{n\to\infty} \int |f_n| - |f_n - f| = \lim_{n\to\infty} \int |f_n| - \int |f_n - f|$$

Since $\int |f_n| \to \int |f|$,

$$0 = \lim_{n\to\infty} \left(\int |f_n| - \int |f_n - f| \right) + \lim_{n\to\infty} \left(-\int |f_n| \right) = \lim_{n\to\infty} -\int |f_n - f|.$$

(8)

(a) Define $f_{1,n}(x) = \frac{n}{x}\sin\left(\frac{x}{n}\right)$, $f_{2,n}(x) = \frac{1}{(1+|x|)^2}$

Both of these functions are cts, hence Borel measurable. $1_{[0,+\infty)}(x)$ is a simple fn, hence Borel measurable. Thus, the product of these three functions $f_n(x)$ is Borel measurable.

All Borel measurable functions are Lebesgue measurable.

(b) First, note that $\left|\frac{\sin(x)}{x}\right| \leq 1$.

Thus $|f_n(x)| \leq g(x) := \frac{1}{(1+x)^2} 1_{[0,+\infty)}(x)$.

Note that $g \in L^1(\lambda)$, since

$$\int g(x)\,d\lambda(x) = \int_{[0,+\infty)} \frac{1}{(1+x)^2} \overset{\text{MCT}}{=} \lim_{b\to+\infty} \int_{[0,b]} \frac{1}{(1+x)^2}\,d\lambda(x)$$

corresp w/ Riemann integral

$$= \lim_{b\to\infty} (-1)(1+x)^{-1}\Big|_0^b$$

$$= \lim_{b\to\infty} \left(1 - \frac{1}{1+b}\right)$$

$$= 1$$

Thus, by DCT,

by above computation for $g(x)$

$$\lim_{n\to\infty} \int f_n\,d\lambda = \int \lim_{n\to\infty} f_n\,d\lambda = \int_{[0,+\infty)} \frac{1}{(1+x)^2} = 1.$$

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(a) Note that, for $\varepsilon > 0$,

$$\mu(\{x : |f_n(x) + g_n(x) - f(x) - g(x)| \geq \varepsilon\})$$

$$\leq \mu(\{x : |f_n(x) - f(x)| + |g_n(x) - g(x)| \geq \varepsilon\})$$

$$\leq \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon/2\}) + \mu(\{x : |g_n(x) - g(x)| \geq \varepsilon/2\})$$

$\to 0$, since $f_n \to f$ and $g_n \to g$ in measure.
Thus $f_n + g_n \to f + g$ in measure.

(b) Note that, for $\varepsilon > 0$,

$$\mu(\{x : |f_n(x) g_n(x) - f(x) g(x)| \geq \varepsilon\})$$

$$\leq \mu(\{x : |f_n(x) g_n(x) - f(x) g_n(x)| + |f(x) g_n(x) - f(x) g(x)| \geq \varepsilon\})$$

$$\leq \underbrace{\mu(\{x : |f_n(x) - f(x)||g_n(x)| \geq \varepsilon/2\})}_{\mathrm{I}} + \underbrace{\mu(\{x : |f(x)||g_n(x) - g(x)| \geq \varepsilon/2\})}_{\mathrm{II}}$$

First, we consider I. For $M \geq 1$,

$$\text{I} \leq \mu(\{x : (|g(x)|+1)|f_n(x)-f(x)| \geq \varepsilon\}$$
$$+ \mu(\{x : |g_n(x)| \geq |g(x)|+1\}$$
$$\leq \underbrace{\mu(\{x : M|f_n(x)-f(x)| \geq \varepsilon\}}_{\text{Ia}}$$
$$+ \underbrace{\mu(\{x : |g(x)| \geq M-1\}}_{\text{Ib}} + \underbrace{\mu(\{x : |g_n(x)| \geq |g(x)|+1\})}_{\text{Ic}}$$

For Ib, define $E_M = \{x : |g(x)| \geq M-1\}$. Since $\mu(E_1) = \mu(X) < +\infty$, by continuity from above, $\lim_{M\to\infty} \mu(E_M) = \mu(\bigcap_{M=1}^{\infty} E_M) = \mu(\emptyset) = 0$, since g is real valued. Thus Ib $\xrightarrow{M\to+\infty} 0$, and we may choose $M \geq 1$ so that this term is arbitrarily small.

For any fixed $M \geq 1$,

Ia $\xrightarrow{n\to+\infty} 0$ since $f_n \to f$ in measure

Ic $\xrightarrow{n\to+\infty} 0$ since $g_n \to g$ in measure.

Thus, I $\xrightarrow{n\to+\infty} 0$.

Finally, we consider II. Again, for $M \geq 1$,

$$\text{II} \leq \mu(\{x : M|g_n(x) - g(x)| \geq \varepsilon\}) + \mu(\{x : |f(x)| > M\})$$

As in Ib above, we may chose $M \geq 1$ so that the second term is arbitrarily small. For any such M, the first term converges to zero as $n \to +\infty$, since $g_n \to g$ in measure. Thus, $\text{II} \xrightarrow{n \to +\infty} 0$.

This shows $f_n g_n \to fg$ in measure.

(c)

Consider the measure space $(\mathbb{R}, \mathcal{M}_{\lambda^*}, \lambda)$. Define $f_n = \frac{1}{n} 1_{[0,n]}$ and $g_n(x) = g(x) = x$. Then, as shown in class, $f_n \to 0$ in measure, and, by definition, $g_n(x) \to g(x)$ in measure. However,

$$\lambda(\{x : |g_n(x) f_n(x)| \geq \tfrac{1}{2}\})$$

$$= \lambda(\{x : \tfrac{x}{n} 1_{[0,n]}(x) \geq \tfrac{1}{2}\})$$

$$\geq \lambda([\tfrac{n}{2}, n]) = \frac{n}{2},$$

which does not converge to zero as $n \to +\infty$.

Thus $f_n g_n \not\to fg$ in measure.

Final Exam Solutions

(1)

(a) $\phi'(s) = \frac{1+s-s}{(1+s)^2} = \frac{1}{(1+s)^2} \geq 0 \Rightarrow \phi$ is nondecreasing

$\phi(s+t) = \frac{s+t}{1+s+t} = \frac{s}{1+s+t} + \frac{t}{1+s+t} \leq \frac{s}{1+s} + \frac{t}{1+t} = \phi(s)+\phi(t)$

$\phi(s) = 0 \Leftrightarrow \frac{s}{1+s} = 0 \Leftrightarrow s = 0$, since $s \geq 0 \Rightarrow 1+s > 0$.

(b) $\rho(f,g) = 0$

$\Leftrightarrow \underbrace{\frac{|f-g|}{1+|f-g|}}_{\psi :=} = 0$ a.e. , since ψ is a nonneg, meas fn.

$\Leftrightarrow \phi(|f(x)-g(x)|) = 0$ a.e.

$\Leftrightarrow |f(x)-g(x)| = 0$ a.e.

$\Leftrightarrow f = g$ a.e.

$\rho(f,g) = \int \phi(|f-g|)d\mu = \int \phi(|f-h+h-g|)d\mu$

ϕ nondecreasing

$\leq \int \phi(|f-h| + |h-g|)d\mu$