Math 201a: Homework 7

Due Sunday, December 8th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice questions will appear on each exam. *All answers should be justified with either a proof or a counterexample.*

Practice Final Exam:

Question 1*

Consider a measure space $(X, \mathcal{M}, \mu)$ and a nonnegative measurable function f satisfying $\int f d\mu = c$ for $0 < c < +\infty$. Let log denote the natural logarithm, and recall the inequality

$$1 + x^r \leq (1+x)^r \leq e^{rx} \text{ for all } x \geq 0, r \geq 1.$$

Prove that

$$\lim_{n \to +\infty} \int n \log(1 + (f/n)^\alpha) d\mu = \begin{cases} +\infty & \text{if } 0 < \alpha < 1, \\ c & \text{if } \alpha = 1, \\ 0 & \text{if } 1 < \alpha < +\infty. \end{cases}$$

You are also welcome to use any standard tools you know from Calculus (e.g. L'Hopital's Rule) to find the limit $\lim_{n \to +\infty} n \log(1 + (y/n)^\alpha)$ for $y \geq 0$.

Question 2*

Consider the measure spaces $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \lambda)$ and $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \nu)$, where λ is Lebesgue measure and ν is the counting measure, that is, for any $A \in \mathcal{B}_\mathbb{R}$,

$$\nu(A) = \text{ the number of elements in } A.$$

Let $D = \{(x,x) : x \in [0,1]\}$.

(a) For fixed $x \in \mathbb{R}$, why is $y \mapsto 1_D(x,y)$ measurable?

(b) Prove that the following two integrals are not equal: $\int \int 1_D d\lambda \, d\nu$ and $\int \int 1_D d\nu \, d\lambda$.

(c) Why does part (b) not violate Tonelli's theorem? Make sure to rigorously justify your answer.

Question 3

Parts (a) and (b) are not related.

(a) Consider the measure space $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \lambda)$, where λ is Lebesgue measure, as well as the product space $(\mathbb{R}^2, \mathcal{B}_\mathbb{R} \otimes \mathcal{B}_\mathbb{R}, \lambda \otimes \lambda)$. Suppose $f : \mathbb{R} \to [0, +\infty)$ is Borel measurable. Prove that $A := \{(x,y) : 0 < y < f(x)\} \subseteq \mathbb{R}^2$ is Borel measurable and $\lambda \otimes \lambda(A) = \int f d\lambda$.

(b) Now, consider the measure space $(\mathbb{R}, \mathcal{M}_{\lambda^*}, \lambda)$ of Lebesgue measurable sets on the real line. On the homework, we showed that Lebesgue measure on this σ-algebra is *complete*, that is, every subset of a set of measure zero must be measurable. Show that the product space $(\mathbb{R}^2, \mathcal{M}_{\lambda^*} \otimes \mathcal{M}_{\lambda^*}, \lambda \otimes \lambda)$ is *not* complete by showing that there exists a set $A \times B$ in $\mathcal{M}_{\lambda^*} \otimes \mathcal{M}_{\lambda^*}$ with $\lambda \otimes \lambda(A \times B) = 0$, such that $A \times B$ contains a set $E \times F \notin \mathcal{M}_{\lambda^*} \otimes \mathcal{M}_{\lambda^*}$.

Question 4*

Let μ^* be an outer measure on a nonempty set X. Prove that A is μ^*-measurable if and only if, for all $B \subseteq A$ and $C \subseteq A^c$, $\mu^*(B \cup C) = \mu^*(B) + \mu^*(C)$.

Extra problems

Question 5

Consider measurable spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$. Recall that a set E is a *rectangle* if $E = A \times B$ for $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Consider the collection of sets

$$\mathcal{A} := \left\{ \cup_{i=1}^n E_i : E_i \text{ are disjoint rectangles for all } i = 1, \dots, n, \quad n \in \mathbb{N} \right\}.$$

Prove that $\mathcal{A}$ is an algebra.

Question 6*

Let $(X_i, \mathcal{M}_i)_{i=1}^3$ be measurable spaces. Suppose μ_i is a σ-finite measure on $\mathcal{M}_i$ for all $i = 1, \dots, 3$. Prove that $(\mu_1 \otimes \mu_2) \otimes \mu_3 = \mu_1 \otimes (\mu_2 \otimes \mu_3)$.

As a consequence of the above fact, we see that taking the product of two σ-finite measures is associative. Thus, for general $n \in \mathbb{N}$, we may *define* the product measure $\mu_1 \otimes \cdots \otimes \mu_n$ as the measure obtained by taking any combination of pairwise products of the μ_i's.

Question 7*

Let $E = [0,1] \times [0,1]$. Investigate the existence and equality of $\int_E f d\lambda^2$, $\int_0^1 f(x,y) d\lambda(x) d\lambda(y)$, and $\int_0^1 f(x,y) d\lambda(y) d\lambda(x)$ for the following choices of f:

(a) $f(x,y) = (x^2 - y^2)(x^2 + y^2)^{-2}$;

(b) $f(x,y) = (1 - xy)^{-a}$ for $a > 0$.

Question 8

For $x \in \mathbb{R}$ and $a > 0$, compute $\frac{d}{dx} a^x$.

(This is just making sure you remember an important Calculus fact that you will use in Q9.)

Question 9

Consider a measure space $(X, \mathcal{M}, \mu)$ where $\mu(X) = 1$.

Recall that a function $F : (0, +\infty) \to \mathbb{R}$ is *convex* if $F(\lambda s + (1-\lambda)t) \leq \lambda F(s) + (1-\lambda) F(t)$ for all $s, t \in (0, +\infty)$ and $\lambda \in (0,1)$. It is a general fact about convex functions that, for any $t_0 \in (0, +\infty)$, there exists $\beta_0 \in \mathbb{R}$ so that

$$F(t) - F(t_0) \geq \beta_0 (t - t_0) \text{ for all } t \in (0, +\infty).$$

(i) Suppose $f \in L^1(\mu)$ is nonnegative and $\log(f) \in L^1(\mu)$. Prove that

$$\lim_{p \to 0^+} \int \frac{f^p - 1}{p} d\mu = \int \log(f) d\mu.$$

Hint: You may use without proof the fact that, for all $t > 0$ and $p \in (0,1)$,

$$\frac{|t^p - 1|}{p} \leq |t - 1| + |\log(t)|.$$

(ii) Suppose $g : X \to (0, +\infty)$ is integrable and $F : (0, +\infty) \to \mathbb{R}$ is convex. Prove that

$$F\left(\int g d\mu\right) \leq \int F \circ g d\mu.$$

Hint: Explain why you can use the fact above with $t_0 = \int g d\mu$ and $t = g(x)$.

(iii) Apply parts (i) and (ii) to show that $\lim_{p \to 0^+} \left(\int f^p d\mu\right)^{1/p} = \exp\left(\int \log(f) d\mu\right)$.

Hint: Explain why it is sufficient to show the result when you take logarithms of both sides. Use part (i) and the inequality $\log(x) \leq x - 1$ for all $x \geq 0$ for an upper bound. Use part (ii) for a lower bound, with the convex function $F(s) = -\log(s)$.