Homework 7 Solutions

By L'Hopital's rule, $\forall \alpha > 0, y > 0$,

① $$\lim_{n\to\infty} \frac{\log(1+(y/n)^\alpha)}{1/n} = \lim_{n\to\infty} \frac{\alpha (y/n)^{\alpha-1}(-y)n^{-2}}{-n^{-2}(1+(y/n)^\alpha)} = \begin{cases} +\infty & 0<\alpha<1 \\ y & \alpha=1 \\ 0 & \alpha>1 \end{cases}$$

First, suppose $0<\alpha<1$.

Since f is nonnegative, by Fatou's Lemma,

$$\liminf_{n\to\infty} \int n \log(1+(f/n)^\alpha) d\mu$$

$$\geq \int \liminf_{n\to\infty} n\log(1+(f/n)^\alpha) d\mu$$

$$= +\infty,$$

since $\int f d\mu = c > 0$ implies $f(x) > 0$ for μ-a.e. x.

Next, suppose $\alpha \geq 1$. ~~Fatou's Lemma again implies $\liminf_{n\to\infty} \int n \log(1+(f/n)^\alpha) d\mu \geq \int f d\mu = c$.~~

~~On the other hand~~

By the given inequality and monotonicity of log,

$$0 \leq n\log(1+(f/n)^\alpha) \leq n\alpha(f/n) = \alpha f$$

Thus, ~~thus~~ by the Dominated Convergence Thm,

$$\lim_{n\to\infty} \int n\log(1+(f/n)^\alpha) d\mu = \int \lim_{n\to\infty} n\log(1+(f/n)^\alpha) d\mu$$

$$= \begin{cases} \int f d\mu & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1. \end{cases}$$

(2)

(a) For any measurable spaces $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$, if a real valued function f is $\mathcal{M} \otimes \mathcal{N}$ measurable, then for any x, the function $g(y) = f(x,y)$ is measurable, since for any Borel set B, $g^{-1}(B) = \{ y : g(y) \in B \} = \{ y : f(x,y) \in B\}$

$$= (\{ (x,y) : f(x,y) \in B\})_x$$

and we showed in class that the x section of any measurable set is measurable.

(b) $\int\int 1_D \, d\lambda \, d\nu = \int \lambda(D^y) \, d\nu(y) = 0$

Since

$$\lambda(D^y) = \begin{cases} \lambda(\{y\}) & \text{if } y \in [0,1] \\ \lambda(\emptyset) & y \notin [0,1] \end{cases} = 0.$$

$$\int\int 1_D \, d\nu \, d\lambda = \int \nu(D_x) \, d\lambda(x) = \int 1_{[0,1]}(x) \, d\lambda(x) = 1$$

(c) ν is not σ-finite since $\mathbb{R}$ is uncountable, hence can't be written as a countable union of finite sets

(4) Assume A is μ^*-measurable.

Fix $B \subseteq A$, $C \subseteq A^c$. Then

$$\mu^*(B \cup C) = \mu^*((B \cup C) \cap A) + \mu^*((B \cup C) \cap A^c)$$
$$= \mu^*(B) + \mu^*(C).$$

Assume the condition in the problem holds. Fix $E \subseteq X$.

$$\mu^*(E) = \mu^*(\underbrace{(E \cap A)}_{B} \cup \underbrace{(E \cap A^c)}_{C})$$
$$= \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Thus A is μ^* meas.

⑥ By uniqueness theorem from lecture, it suffices to show LHS and RHS coincide on the algebra of disjoint unions of rectangles and the measure on the LHS is strongly σ-finite.

Since each μ_i is σ-finite $\exists \{E^i_j\}_{j=1}^{\infty}$ s.t. $\mu_i(E^i_j) < +\infty$ $\forall i,j$ and $\bigcup_j E^i_j = X^i$. WLOG $j \mapsto E^i_j$ are increasing, so, for all j,

$$(\mu_1 \otimes \mu_2) \otimes \mu_3 (E^1_j \times E^2_j \times E^3_j) = (\mu_1 \otimes \mu_2)(E^1_j \times E^2_j)\, \mu_3(E^3_j)$$
$$= \mu_1(E^1_j)\, \mu_2(E^2_j)\, \mu_3(E^3_j) < +\infty$$

and $\bigcup_{j=1}^{\infty} E^1_j \times E^2_j \times E^3_j = X^1 \times X^2 \times X^3$. This shows the LHS is strongly σ-finite.

Finally, if $\bigcup_{i=1}^{N} R^1_i \times R^2_i \times R^3_i$ is a finite disj union of rectangles,

$$(\mu_1 \otimes \mu_2) \otimes \mu_3 \left(\bigcup_{i=1}^{N} R^1_i \times R^2_i \times R^3_i \right)$$

$$= \sum_{i=1}^{N} (\mu_1 \otimes \mu_2) \otimes \mu_3 (R^1_i \times R^2_i \times R^3_i)$$

$$= \sum_{i=1}^{N} \mu_1(R_i^1)\,\mu_2(R_i^2)\,\mu_3(R_i^3)$$

$$= \sum_{i=1}^{N} \mu_1 \otimes (\mu_2 \otimes \mu_3)(R_i^1 \times R_i^2 \times R_i^3)$$

$$= \mu_1 \otimes (\mu_2 \otimes \mu_3)\left(\bigcup_{i=1}^{N} R_i^1 \times R_i^2 \times R_i^3\right).$$

Question 7

All functions in this problem are piecewise cts, hence Borel measurable.

(a)

By Tonelli's Theorem,

$$\int_E (f)_- \, d\lambda\otimes\lambda = -\int_{E\cap\{(x,y):x<y\}} (x^2-y^2)(x^2+y^2)^{-2} \, d\lambda\otimes\lambda$$
$$= -\int_{(0,1)} \int_{[0,y)} (x^2-y^2)(x^2+y^2)^{-2} \, d\lambda(x) \, d\lambda(y)$$

For $y>0$, the equivalence of Riemann and Lebesgue integrals ensures

$$\int_{[0,y)} (x^2-y^2)(x^2+y^2)^{-2} \, d\lambda(x) = \int_0^y (x^2-y^2)(x^2+y^2)^{-2} \, dx$$

$$\underset{\text{Mathematica}}{=} \frac{-1}{2y}$$

Thus $-\int_{(0,1)} \frac{-1}{2y} dy = +\infty$, so $\int_E (f)_- \, d\lambda\otimes\lambda = +\infty$.

Likewise,

$$\int_E (f)_+ \, d\lambda\otimes\lambda = \int_{E\cap\{(x,y):x>y\}} (x^2-y^2)(x^2+y^2)^{-2} \, d\lambda\otimes\lambda$$
$$= \int_{(0,1)} \int_{[y,1)} (x^2-y^2)(x^2+y^2)^{-2} \, dx \, d\lambda(y)$$
$$= \int_{(0,1)} \frac{(y-1)^2}{2y(1+y^2)} \, d\lambda(y)$$

Monotone convergence theorem

$$= \lim_{b\downarrow 0} \int_{(b,1)} \frac{(y-1)^2}{2y(1+y^2)} \, d\lambda(y)$$

Equivalence of Riemann and Lebesgue

$$= \lim_{b\downarrow 0} \int_b^1 \frac{(y-1)^2}{2y(1+y^2)} \, dy = +\infty$$

Thus $\int_E f \, d\lambda \otimes \lambda$ does not exist.

On the other hand, for $y \in [0,1]$, arguing as before,

$$\int_{[0,1]} (f(x,y))_+ \, d\lambda(x) = \int_y^1 (x^2-y^2)(x^2+y^2)^{-2} \, dx = \frac{(y-1)^2}{2y(1+y^2)}$$

$$\int_{[0,1]} (f(x,y))_- \, d\lambda(x) = \frac{1}{2y}$$

so $$\int_0^1 f(x,y) \, d\lambda(x) = \frac{(y-1)^2}{2y(1+y^2)} - \frac{1}{2y} = \frac{1}{2y}\left(\frac{(y-1)^2-(1+y^2)}{(1+y^2)}\right)$$

$$= \frac{1}{2y} \frac{-2y}{(1+y^2)} = \frac{-1}{(1+y^2)}$$

Thus,

Equiv of Riemann and Lebesgue

$$\int_{[0,1]} \int_{[0,1]} f(x,y) \, d\lambda(x) \, d\lambda(y) = \int_0^1 \frac{-1}{(1+y^2)} \, dy = \frac{-\pi}{4}$$

mathematica

Finally, ~~$x \in [0,1]$~~ since $f(x,y) = -f(y,x)$,

$$\int_{[0,1]} \int_{[0,1]} f(x,y) \, d\lambda(y) \, d\lambda(x) = \int_{[0,1]} \int_{[0,1]} -f(y,x) \, d\lambda(y) \, d\lambda(x) = \frac{\pi}{4}$$

(b) Note that $f \geq 0$ on E. Thus, all integrals exist.
By Tonelli,

$$\int_E f \, d\lambda\otimes\lambda = \int_0^1 \int_0^1 f(x,y) \, d\lambda(x) \, d\lambda(y)$$

$$= \int_0^1 \int_0^1 f(x,y) \, d\lambda(y) \, d\lambda(x)$$