Lecture 10

Recall:

Thm (Beppo-Levi): Given $\{f_n\}_{n=1}^{\infty} : X \to [0, +\infty]$ measurable functions, then

$$\sum_{n=1}^{\infty} \int f_n \, d\mu = \int \sum_{n=1}^{\infty} f_n \, d\mu.$$

Thm (Fatou's Lemma): Given $\{f_n\}_{n=1}^{\infty} : X \to [0, +\infty]$ measurable,

$$\liminf_{n\to\infty} \int f_n \, d\mu \geq \int \liminf_{n\to\infty} f_n \, d\mu.$$

Prop: Given $f: X \to [0,+\infty]$ meas,

$$\int f\,d\mu = 0 \iff f = 0 \quad \mu\text{-a.e.}$$

Integration of Real-Valued Functions

Measure space $(X, \mathcal{M}, \mu)$.

Given $f: X \to \overline{\mathbb{R}}$, define

"positive part"

$$\left.\begin{aligned} f_+ &= f \vee 0 \\ f_- &= (-f) \vee 0 \end{aligned}\right\} \begin{aligned} f &= f_+ - f_- \\ |f| &= f_+ + f_- \end{aligned}$$

"negative part"

Def: Given $f: X \to \overline{\mathbb{R}}$ meas, if either $\int f_+ d\mu$ or $\int f_- d\mu$ is finite,

$$\int f d\mu := \int f_+ d\mu - \int f_- d\mu$$

$\Leftrightarrow$ $\int f_+ + f_- d\mu < +\infty \Leftrightarrow \int |f| d\mu < +\infty$

If both $\int f_+ d\mu$ and $\int f_- d\mu$ are finite, we say f is integrable and write $f \in L^1(\mu)$.

Prop: $L^1(\mu)$ is a real vector space and

$$f \mapsto \int f d\mu$$

is a linear functional on $L^1(\mu)$.

Pf:
To see that $L^1(\mu)$ is a real vector space, fix $a, b \in \mathbb{R}$ and $f, g \in L^1(\mu)$. We must show $af + bg \in L^1(\mu)$.

- $af + bg$ is measurable
- $|af + bg|(x) \leq |a||f|(x) + |b||g|(x)$ $\forall x \in X$
- hence $\int |af + bg| d\mu \leq \int |a||f| + |b||g| d\mu$

$\ldots = |a| \int |f| d\mu + |b| \int |g| d\mu < +\infty$

Thus, $af + bg \in L^1(\mu)$.

To see integration is a linear functional, Fix $f \in L^1(\mu)$ and $a \geq 0$. Then

$$\int af\,d\mu = \int af_+\,d\mu - \int af_-\,d\mu$$
$$= a\int f_+\,d\mu - a\int f_-\,d\mu$$
$$= a\int f\,d\mu$$

Furthermore, the same computation for $-f$ shows that

$$\int af\,d\mu = a\int f\,d\mu \quad \forall a \in \mathbb{R}.$$

Finally, for any $f, g \in L^1(\mu)$,

$$\int f+g\,d\mu = \int (f+g)_+\,d\mu - \int (f+g)_-\,d\mu$$

$$\stackrel{1}{=} \int f_+\,d\mu + \int g_+\,d\mu - \int f_-\,d\mu - \int g_-\,d\mu$$

$$= \int f\,d\mu + \int g\,d\mu$$

Justification of $\perp$:

$(f+g)_+ - (f+g)_- = f_+ - f_- + g_+ - g_-$

$(f+g)_+ + f_- + g_- = (f+g)_- + f_+ + g_+$

$\downarrow$ Beppo Levi

$$\int (f+g)_+ d\mu + \int f_- d\mu + \int g_- d\mu = \int (f+g)_- d\mu + \int f_+ d\mu + \int g_+ d\mu$$

Rearranging gives $\perp$. $\square$

Prop: If $f \in L^1(\mu)$, then

$$\left|\int f \, d\mu\right| \leq \int |f| \, d\mu.$$

Pf: $\text{LHS} = \left|\int f_+ d\mu - \int f_- d\mu\right|$

$\leq \int f_+ d\mu + \int f_- d\mu = \int |f| d\mu$ $\square$

Goal: Wish we could show $L^1(\mu)$ is a metric space (in fact, a normed vector space).

Guess for metric:

$$\|f-g\|_{L^1(\mu)} := \int |f-g| \, d\mu$$

Problem: this is degenerate

Ex: $f = 1_{\{0\}}$, $g = 0$

then $\|f-g\|_{L^1(\lambda)} = \int |f| \, d\lambda = 0$,

but $f \neq g$.

Cor: If $f, g \in L^1(\mu)$, then

$$\int |f-g| \, d\mu = 0 \iff f = g \quad \mu\text{-a.e.}$$

Pf: This follows from previous proposition.

Moral:

- If you modify an integrable function on a null set, it doesn't change the integral:

$$\left|\int f\,d\mu - \int g\,d\mu\right| \le \int |f-g|\,d\mu$$

- Even if a function f is only defined μ-a.e., $\int f\,d\mu$ is still uniquely determined.

This motivates a modified defn of $\mathcal{L}^1(\mu)$...

Def:

$L^1(\mu) = \{ f : X \to \overline{\mathbb{R}}$ measurable, $\int |f| d\mu < +\infty \} / \sim$

where $f \sim g$ iff $f = g$ μ-a.e.

Rmk: By abuse of notation, let $f \in L^1(\mu)$ denote...

1. the equivalence class
2. a representative of the equiv class
3. a representative that is only defined μ-a.e..

Prop: $\|f\|_{L^1(\mu)} := \int |f| d\mu$ is a norm on $L^1(\mu)$.

Pf:
nondegenerate ✓
triangle ineq ✓
positive homogeneity ✓ $\square$

Previously, we used montonicity to interchange limits and integrals of nonneg fns. Now we use boundedness to do so for real valued fns.

MAJOR THM 6

Thm: (Dominated Convergence)
Given $\{f_n\}_{n=1}^{\infty} \subseteq L^1(\mu)$
s.t. $\lim_{n\to\infty} f_n$ exists μ-a.e., if $\exists\, g \in L^1(\mu)$ s.t. $\forall n \in \mathbb{N}$, $|f_n| \leq g$ μ-a.e., then

$$\lim_{n\to\infty}\int f_n\,d\mu = \int \lim_{n\to\infty} f_n\,d\mu.$$

<u>Rmk:</u>

Let $E_n := \{x : |f_n|(x) > g(x)\}$.

By hypothesis, $\mu(E_n) = 0 \ \forall n \in \mathbb{N}$.

Let $E = \bigcup_{n\in\mathbb{N}} E_n$. Then

$$\mu(E) \le \sum_{n\in\mathbb{N}} \mu(E_n) = 0.$$

<u>Rmk:</u>

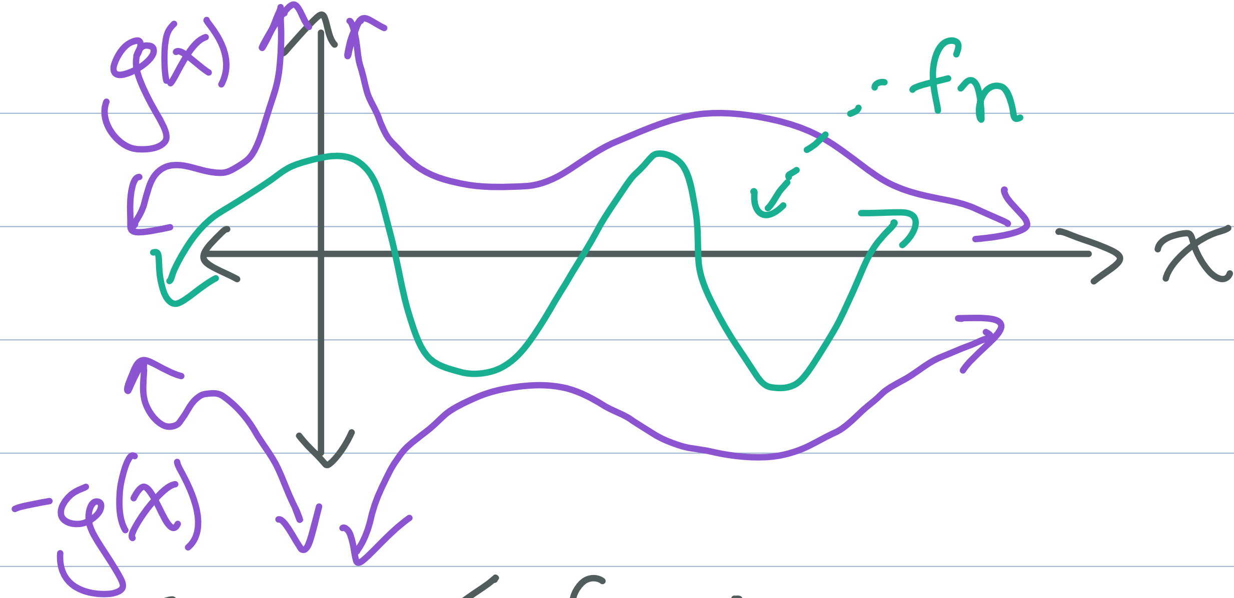
i.e. $\exists M \ge 0$ s.t. $|g(x)| \le M \ \forall x$

$g \in L^1(\lambda) \not\Leftrightarrow g$ pointwise bdd

$$g(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \in L^1(\lambda)$$

$$g(x) = 1 \notin L^1(\lambda).$$

Pf:



$$|f_n| \le g \iff -g \le f_n \le g$$

Since $\lim_{n\to\infty} f_n$ exists μ-a.e. and $|f_n| \le g$ μ-a.e. $\forall n \in \mathbb{N}$, we have

$$\left|\lim_{n\to\infty} f_n\right| \le g \quad \mu\text{-a.e.}$$

Thus $\lim_{n\to\infty} f_n \in L^1(\mu)$.

Since $g - f_n \geq 0$ and $g + f_n \geq 0$ μ-a.e., by Fatou's lemma,

$\liminf_{n\to\infty}$

$$
\begin{aligned}
\int g + \underline{\lim} \int f_n &= \underline{\lim} \int g + \int f_n \\
&= \underline{\lim} \int g + f_n \\
&\geq \int \underline{\lim}\, g + f_n \\
&= \int g + \underline{\lim}\, f_n \\
&= \int g + \int \underline{\lim}\, f_n
\end{aligned}
$$

dropping $d\mu$ for notational simplicity

$$
\begin{aligned}
\int g - \overline{\lim} \int f_n &= \underline{\lim} \int g - \int f_n \\
&= \underline{\lim} \int g - f_n \\
&\geq \int \underline{\lim}\, g - f_n \\
&= \int g - \overline{\lim}\, f_n \\
&= \int g - \int \overline{\lim}\, f_n
\end{aligned}
$$

Since $g \in L^1(\mu) \Rightarrow \int g < +\infty$, subtracting $\int g$ from both sides..

$$\int \underline{\lim}\, f_n \leq \underline{\lim} \int f_n$$

limfn exists
μ-a.e.

$$\|$$

$$\overline{\lim} \int f_n \leq \int \overline{\lim}\, f_n$$

Thus, equality holds throughout, which gives the result. $\square$

We will now apply DCT to identify two useful subsets of functions that are <u>dense</u> in $L^1(\mu)$.

↙ MAJOR THM 7

<u>Thm</u>: For any measure space $(X, \mathcal{M}, \mu)$, simple functions are dense in $L^1(\mu)$.

If μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$, the following are dense in $L^1(\mu)$:

- simple fns of the form

$$\xi = \sum_{j=1}^{N} a_i \mathbb{1}_{F_j}, \quad F_j = \bigcup_{i=1}^{m_j} I_{ij}$$

for disjoint open intervals $\{I_{ij}\}_{i=1}^{m_j}$.

- $C_c(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : f \text{ cts and } \overline{\{x : f(x) \neq 0\}} \text{ is compact}\}$.

Pf of Thm: Fix $f \in L^1(\mu)$.

Since f_+, f_- are nonneg, meas,
$\exists\ \varphi_n \nearrow f_+,\ s_n \nearrow f_-$.

simple functions

Furthermore,

$$\underbrace{|(\varphi_n - s_n) - f|}_{\text{sequence in } L^1(\mu) \text{ that converges to zero pointwise}} \le \varphi_n + s_n + |f| \le \underbrace{2|f|}_{\text{dominating function}}$$

By DCT,

$$\lim_{n\to\infty} \int |(\varphi_n - s_n) - f|\,d\mu$$
$$= \int \lim_{n\to\infty} |(\varphi_n - s_n) - f|\,d\mu$$
$$= \int 0\,d\mu$$
$$= 0.$$

This shows simple fns are dense in $L^1(\mu)$.

Finish next time...