Lecture 11

Recall:

Prop: If $f \in L^1(\mu)$, then
$$\left|\int f \, d\mu\right| \le \int |f| \, d\mu.$$

Cor: If $f, g \in L^1(\mu)$, then
$$\int |f-g| \, d\mu = 0 \iff f = g \ \mu\text{-a.e.}$$

Def:
$$L^1(\mu) = \left\{ f : X \to \overline{\mathbb{R}} \text{ measurable}, \int |f| \, d\mu < +\infty \right\} / \sim$$
where $f \sim g$ iff $f = g$ μ-a.e.

Rmk: By abuse of notation, let $f \in L^1(\mu)$ denote...
1. the equivalence class
2. a representative of the equiv class
3. a representative that is only defined μ-a.e..

Prop: $\|f\|_{L^1(\mu)} := \int |f| d\mu$ is a norm on $L^1(\mu)$.

MAJOR THM 6

Thm: (Dominated Convergence)
Given $\{f_n\}_{n=1}^{\infty} \subseteq L^1(\mu)$ s.t. $\lim_{n\to\infty} f_n$ exists μ-a.e., if $\exists\, g \in L^1(\mu)$ s.t. $\forall n \in \mathbb{N}$, $|f_n| \le g$ μ-a.e., then

$$\lim_{n\to\infty} \int f_n d\mu = \int \lim_{n\to\infty} f_n d\mu.$$

MAJOR THM 7

Thm: For any measure space $(X, \mathcal{M}, \mu)$, simple functions are dense in $L^1(\mu)$.

If μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$, the following are dense in $L^1(\mu)$:

- simple fns of the form

$$\xi = \sum_{j=1}^{N} a_i \mathbb{1}_{F_j}, \quad F_j = \bigcup_{i=1}^{m_j} I_{ij}$$

for disjoint open intervals $\{I_{ij}\}_{i=1}^{m_j}$.

- $C_c(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : f \text{ cts and } \overline{\{x : f(x) \neq 0\}} \text{ is compact}\}$.

Pf: Fix $f \in L^1(\mu)$.

Last time, we showed that, $\forall \varepsilon > 0$, $\exists\, \phi$ simple, $|\phi| \le |f|$, s.t.

$$\|\phi - f\|_{L^1(\mu)} < \varepsilon.$$

Suppose μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$.

Strategy of argument:

$$\begin{tikzcd}
f \in L^1(\mu) & \phi \text{ simple} \arrow[l, "\text{is close to}"'] & \xi \text{ "really simple"} \arrow[l] \\
 & & g \in C_c(\mathbb{R}) \arrow[u]
\end{tikzcd}$$

Fix ϕ simple. We may express

$$\phi = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j},$$

where $a_j \neq 0 \ \forall j$, $\{E_j\}_{j=1}^{n}$ are disjoint.

By construction, $\forall j = 1, \dots, n$,
$|a_j|\,\mu(E_j) \le \int |\phi|\,d\mu < \int |f|\,d\mu < +\infty$,
so $\mu(E_j) < +\infty \ \forall j$.

Recall from HW 4: For any $E \in \mathcal{M}_{\mu^*}$ with $\mu(E) < +\infty$, $\varepsilon > 0$, $\exists$ disjoint open intervals $\{I^i\}_{i=1}^{m}$ s.t.

$$\mu\Big(E \,\triangle\, \bigcup_{i=1}^{m} I^i\Big) < \varepsilon$$

$A \triangle B = (A \setminus B) \cup (B \setminus A)$

Thus $\forall \varepsilon > 0$, $\exists$ disjoint open intervals $\{I_j^i\}_{i=1}^{m_j}$ s.t.

$$\mu\left(E_j \triangle \bigcup_{i=1}^{m_j} I_j^i\right) < \frac{\varepsilon}{n \max_j |a_j|}.$$

Thus,

$$\left\| \phi - \sum_{j=1}^{n} a_j \mathbb{1}_{\bigcup_{i=1}^{m_j} I_j^i} \right\|_{L^1(\mu)}$$

$$\leq \sum_{j=1}^{n} |a_j| \left\| \mathbb{1}_{E_j} - \mathbb{1}_{\bigcup_{i=1}^{m_j} I_j^i} \right\|_{L^1(\mu)}$$

$$= \sum_{j=1}^{n} |a_j| \mu\left(E_j \triangle \bigcup_{i=1}^{m_j} I_j^i\right)$$

$$< \varepsilon.$$

This shows "really simple" functions are dense in $L^1(\mu)$.

Now, fix ξ "really simple";

$$\xi = \sum_{j=1}^{n} a_j \mathbb{1}_{\bigcup_{i=1}^{m_j} I_j^i}$$

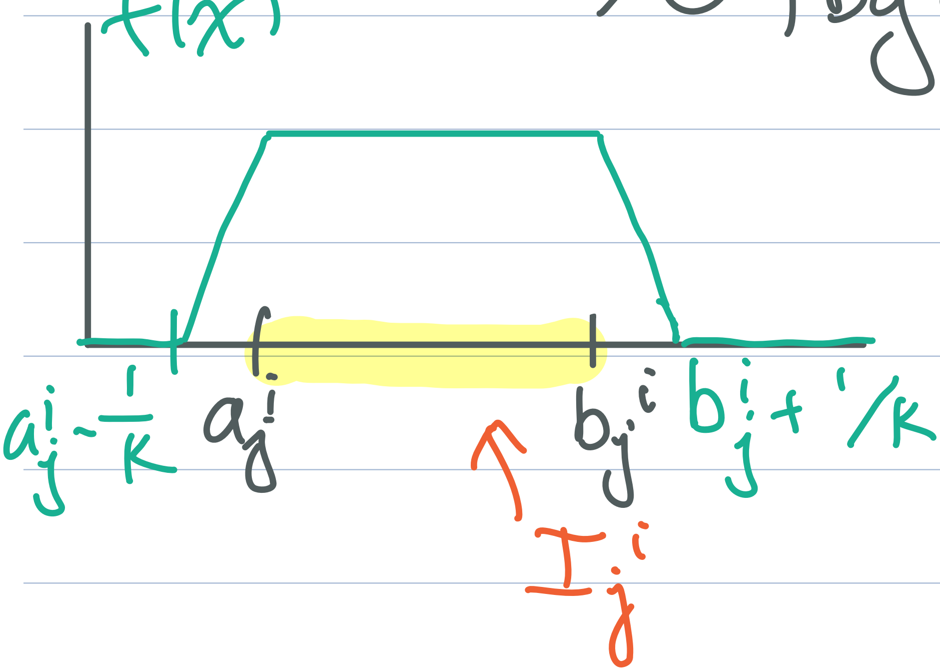
we will show it can be approximated by $g \in C_c(\mathbb{R})$. Fix $\varepsilon > 0$.

For any open interval I_j^i, $\exists f_j^i \in C_c$ s.t.

$$\|\mathbb{1}_{I_j^i} - f_j^i\|_{L^1(\mu)} = \int_{(a_j^i - \frac{1}{K}, a_j^i)} f_j^i \, d\mu + \int_{(b_j^i, b_j^i + \frac{1}{K})} f_j^i \, d\mu$$

$$\leq \mu(a_j^i - \tfrac{1}{K}, a_j^i) + \mu(b_j^i, b_j^i + \tfrac{1}{K}) \xrightarrow{K \to \infty} 0$$

, by cty from above, $K \in \mathbb{N}$ since L-S measures are locally bounded.



Thus, we may choose $f_j^i \in C_c(\mathbb{R})$ s.t.

$$\| 1_{I_j^i} - f_j^i \|_{L^1(\mu)} < \frac{\varepsilon}{(n)\max_j |a_j| \max_j m_j}$$

Thus

$$\Big\| \sum_{j=1}^n a_j 1_{\bigcup_{i=1}^{m_j} I_j^i} - \overbrace{\sum_{j=1}^n \sum_{i=1}^{m_j} a_j f_j^i}^{g \in C_c(\mathbb{R})} \Big\|_{L^1(\mu)}$$

$$\le \sum_{j=1}^n |a_j| \, \Big\| \underbrace{1_{\bigcup_{i=1}^{m_j} I_j^i}}_{= \sum_{i=1}^{m_j} 1_{I_j^i}, \text{ since disjoint}} - \sum_{i=1}^{m_j} f_j^i \Big\|_{L^1(\mu)}$$

$$\le \sum_{j=1}^n |a_j| \sum_{i=1}^{m_j} \| 1_{I_j^i} - f_j^i \|_{L^1(\mu)} < \varepsilon \quad \square$$

One more type of result for exchanging "limits" with integrals.

Thm: Fix $a<b$ and consider $f : X \times [a,b] \to \mathbb{R}$. Denote

$$(x,t) \mapsto f(x,t) \in \mathbb{R}.$$

Suppose that let f denote a representative of equiv class

(i) $f(\cdot,t) \in L^1(\mu)$ $\forall t \in [a,b]$

(ii) $\dfrac{\partial f}{\partial t}(x,t)$ exists $\forall (x,t) \in X \times [a,b]$

(iii) $\exists\ g \in L^1(\mu)$ s.t.

$$\left| \frac{\partial f}{\partial t}(x,t) \right| \leq g(x) \quad \forall (x,t) \in X \times [a,b]$$

Then $t \mapsto \int_X f(x,t)\,d\mu(x)$ is differentiable

and $\frac{d}{dt}\int_X f(x,t)\,d\mu(x) = \int_X \frac{\partial}{\partial t} f(x,t)\,d\mu(x)$.

Pf: Fix $t_0 \in [a,b]$, $\{t_n\}_{n=1}^{\infty} \subseteq [a,b] \setminus \{t_0\}$ with $t_n \to t_0$.

By (ii), $\frac{\partial f}{\partial t}(x,t_0) = \lim_{n\to\infty} \underbrace{\boxed{\frac{f(x,t_n)-f(x,t_0)}{t_n-t_0}}}_{h_n(x)}$,

so $\frac{\partial f}{\partial t}(\cdot,t_0)$ is measurable.

By the Mean Value Theorem + (iii),

$$|h_n(x)| \le \sup_{t\in[a,b]} \left|\frac{\partial f}{\partial t}(x,t)\right| \le g(x)$$

By DCT,

$$\lim_{n\to\infty} \frac{\int f(x,t_n)\,d\mu(x) - \int f(x,t_0)\,d\mu(x)}{t_n - t_0}$$

$\downarrow$ linearity of integral

$$= \lim_{n\to\infty} \int h_n(x)\,d\mu(x)$$

$\downarrow$ DCT

$$= \int \lim_{n\to\infty} h_n(x)\,d\mu(x)$$

$$= \int \frac{\partial f}{\partial t}(x,t_0)\,d\mu(x).$$

Since $t_n \to t_0$ was arbitrary, this gives the result.

Modes of Convergence

$(X, \mathcal{M}, \mu)$ $f_n, f : X \to \overline{\mathbb{R}}$ meas

① uniform convergence $\sup_{x \in X} |f_n(x) - f(x)| \to 0$

$\Downarrow$

② pointwise convergence $f_n(x) \to f(x)\ \forall x \in X$

$\Downarrow$

③ pointwise a.e. conv $f_n(x) \to f(x)$ μ-a.e. $x \in X$

④ L^1 convergence $\|f_n - f\|_{L^1(\mu)} \to 0$

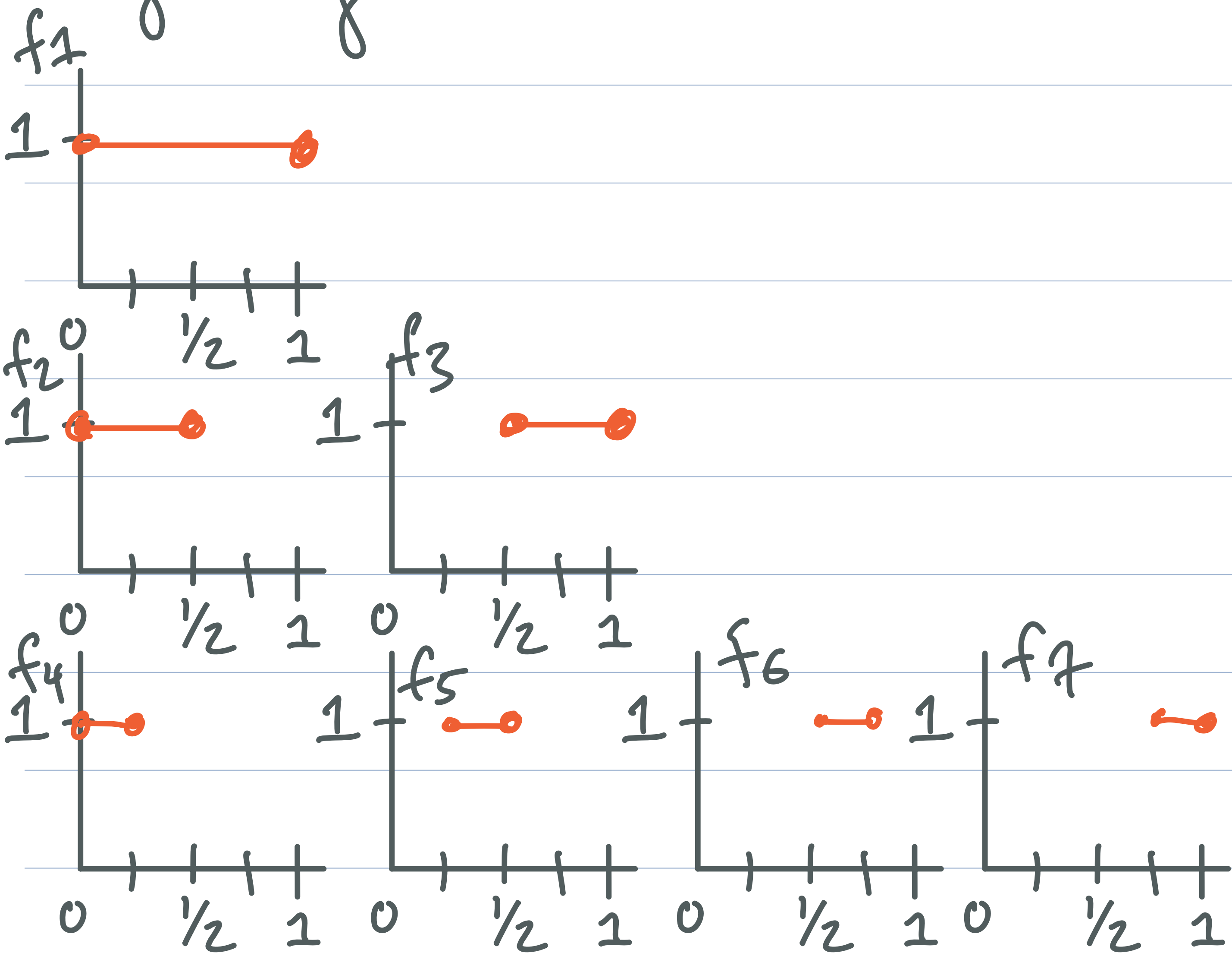
How are ③ + ④ related?

Ex: ① $\not\Rightarrow$ ④

"splat" $f_n = \frac{1}{n} \mathbb{1}_{[0,n]}$, $f = 0$

Ex: ④ $\not\Rightarrow$ ③

"refining wave"



$f = 0$,

$f_n \to f$ in $L^1(\mu)$

$f_n(x) \not\to f(x)$ at <u>any</u> $x \in [0,1]$

With add'l assumptions, something can be salvaged...

- ③ + dominating fn $\Rightarrow$ ④

$\exists\, g \in L^1(\mu)$ s.t. $|f_n(x)| \leq g(x)$ μ-a.e.

- We will now work up to showing ④ $\Rightarrow$ ③ "up to a subsequence"

$\exists\, f_{n_k}$ s.t. $f_{n_k} \to f$ μ-a.e.

Next time ☺