Lecture 12

Recall:

Thm: Fix $a<b$ and consider $f: X \times [a,b] \to \mathbb{R}$. Denote

$$(x,t) \mapsto f(x,t) \in \mathbb{R}$$

Suppose that let f denote a representative of equiv class

(i) $f(\cdot, t) \in L^1(\mu)$ $\forall t \in [a,b]$

(ii) $\frac{\partial f}{\partial t}(x,t)$ exists $\forall (x,t) \in X \times [a,b]$

(iii) $\exists\ g \in L^1(\mu)$ s.t.

$$\left| \frac{\partial f}{\partial t}(x,t) \right| \leq g(x) \quad \forall (x,t) \in X \times [a,b]$$

Then $t \mapsto \int_X f(x,t)\,d\mu(x)$ is differentiable

and $\frac{d}{dt} \int_X f(x,t)\,d\mu(x) = \int_X \frac{\partial}{\partial t} f(x,t)\,d\mu(x)$.

Modes of Convergence

$(X, \mathcal{M}, \mu)$ $f_n, f : X \to \overline{\mathbb{R}}$ meas

1. uniform convergence $\sup_{x \in X} |f_n(x) - f(x)| \to 0$

$\Downarrow$

2. pointwise convergence $f_n(x) \to f(x)$ $\forall x \in X$

$\Downarrow$

3. pointwise a.e. conv $f_n(x) \to f(x)$ μ-a.e. $x \in X$

4. L^1 convergence $\|f_n - f\|_{L^1(\mu)} \to 0$

How are 3 + 4 related?

With add'l assumptions, something can be salvaged...

- 3 + dominating fn $\Rightarrow$ 4

$\exists g \in L^1(\mu)$ s.t. $|f_n(x)| \le g(x)$ μ-a.e.

- We will now work up to showing (4) $\Rightarrow$ (3) "up to a subsequence"

$\exists f_{n_k}$ s.t. $f_{n_k} \to f$ μ-a.e.

not $\overline{\mathbb{R}}$

Def: A sequence of measurable functions $f_n : X \to \mathbb{R}$ <u>converges in measure</u> to a <u>measurable function</u> $f: X \to \mathbb{R}$ if, $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

Likewise, f_n is <u>Cauchy in measure</u> if, $\forall \varepsilon > 0$,

$$\lim_{m,n\to\infty} \mu(\{x : |f_n(x) - f_m(x)| \geq \varepsilon\}) = 0.$$

Rmk: We will show on HW that, if $\mu(X) < +\infty$, then convergence in measure is metrizable.

Rmk: Recall that, on any metric space, if a sequence is convergent, it must be Cauchy.

Rmk: In fact, for all $(X, \mathcal{M}, \mu)$ (even $\mu(X) = +\infty$), f_n convergent in measure $\Rightarrow$ f_n Cauchy in measure.

Thm: Consider $f_n : X \to \mathbb{R}$ meas.

(i) If f_n is Cauchy in measure, then $\exists\, f : X \to \mathbb{R}$ meas. s.t.

(a) $f_n \to f$ in measure

(b) $\exists\, f_{n_k}$ s.t. $f_{n_k} \to f$ μ-a.e.

(ii) If, in addition, $f_n \to g$ in measure, then $f = g$ μ-a.e.

Pf: First, we find our guess for f.

Since f_n is Cauchy in measure, there exists a subsequence f_{n_k} s.t. $\forall\, k \in \mathbb{N}$,

$$\mu\Big(\underbrace{\{x : |\overbrace{f_{n_k}(x)}^{g_k(x)} - \overbrace{f_{n_{k+1}}(x)}^{g_{k+1}(x)}| \geq \tfrac{1}{2^k}\}}_{E_k}\Big) \leq \frac{1}{2^k}$$

By countable subadditivity,

$$\mu\Big(\underbrace{\bigcup_{k=\ell}^{\infty} E_k}_{F_\ell}\Big) \leq \sum_{k=\ell}^{\infty} \mu(E_k) = \frac{1}{2^{\ell-1}}, \ \forall \ell \in \mathbb{N}$$

Note that, if $\overbrace{x \notin F_\ell}^{x \notin E_k \text{ for any } k \geq \ell}$, then for $i \geq j \geq \ell$

$$|g_i(x) - g_j(x)| \leq \sum_{k=j}^{i} |g_k(x) - g_{k+1}(x)|$$

$$\leq \sum_{k=j}^{i} \frac{1}{2^k} \leq \frac{1}{2^{j-1}}$$

Thus, if $\overbrace{x \notin F_\ell \text{ for some } \ell \in \mathbb{N}}^{x \in \bigcup_{\ell=1}^{\infty} F_\ell^c = (F)^c}$, then $\{g_i(x)\}_{i=1}^{\infty} \subseteq \mathbb{R}$ is Cauchy, so it converges to $\lim_{i\to\infty} g_i(x) \in \mathbb{R}$.

Let $F = \bigcap_{\ell=1}^{\infty} F_\ell$

Define $f(x) = \begin{cases} \lim_{i \to \infty} g_i(x) & \text{if } x \in F^c \\ 0 & \text{if } x \in F \end{cases}$

Note that, for any $\ell \in \mathbb{N}$,
$\mu(F) \leq \mu(F_\ell) \leq \frac{1}{2^{\ell-1}} \Rightarrow \mu(F) = 0$.

Therefore, $f_{n_k} = g_k \xrightarrow{k \to \infty} f$ μ-a.e.
This completes the proof of (i)(b).

Now, we show (i)(a).

If $x \notin F_\ell$ and $j \geq \ell$
$|f(x) - g_j(x)| = \lim_{i \to \infty} |g_i(x) - g_j(x)| \leq \frac{1}{2^{j-1}}$.

Therefore if $j \geq l$ and

$|f(x) - g_j(x)| > \frac{1}{2^{j-1}}$

we must have $x \in F_l$.

Thus, $\forall\ \varepsilon > 0$ and $l \in \mathbb{N}$ suff large, s.t. $\varepsilon > \frac{1}{2^{l-1}}$, if $j \geq l$,

$\mu(\{x : |f(x) - g_j(x)| \geq \varepsilon\})$

$\leq \mu(\{x : |f(x) - g_j(x)| \geq \frac{1}{2^{j-1}}\}$

$\leq \mu(F_l) \leq \frac{1}{2^{l-1}}$

Thus, $\lim_{j \to \infty} \mu(\{x : |f(x) - g_j(x)| \geq \varepsilon\}) = 0,$

so $g_j \to f$ in measure.

Finally, for any $\varepsilon > 0$, $j \in \mathbb{N}$,

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\}$$
$$\subseteq \{x : |f_n(x) - g_j(x)| \geq \varepsilon/2\}$$
$$\cup \{x : |f(x) - g_j(x)| \geq \varepsilon/2\}$$

$$\Rightarrow \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\})$$

$$\leq \mu(\{x : |f_n(x) - g_j(x)| \geq \varepsilon/2\})$$
$$+ \mu(\{x : |f(x) - g_j(x)| \geq \varepsilon/2\})$$

Since the RHS $\to 0$ as $n, j \to \infty$, we conclude that $f_n \to f$ in meas.

Finally, we show (ii).
Suppose $f_n \to g$ in measure.
Fix $\varepsilon > 0$. Then,

$$\{x : |g(x) - f(x)| \geq \varepsilon\}$$
$$\subseteq \{x : |f_n(x) - g(x)| \geq \varepsilon/2\}$$
$$\cup \{x : |f_n(x) - f(x)| \geq \varepsilon/2\}$$

Since the measure of RHS $\to 0$ as $n \to +\infty$, $\mu(\{x : |g(x) - f(x)| \geq \varepsilon\}) = 0$.

$$\mu(\{x : |g(x) - f(x)| > 0\})$$

$$\leq \mu\left(\bigcup_{k=1}^{\infty} \{x : |g(x) - f(x)| > \tfrac{1}{k}\}\right)$$

$$\leq \sum_{k=1}^{\infty} \overbrace{\mu(\{x : |g(x) - f(x)| > \tfrac{1}{k}\})}^{0} = 0.$$

Thus $g = f$ μ-a.e.. $\square$

We now apply this to study convergence in $L^1(\mu)$.

Prop:

(a) If f_n is Cauchy in $L^1(\mu)$, then it's Cauchy in measure.

(b) If f_n is convergent in $L^1(\mu)$, then it's convergent in measure.

Pf:

First, we will show (a) $\Rightarrow$ (b). Suppose f_n is convergent in $L^1(\mu)$.

Then, since $L^1(\mu)$ is a metric space, f_n is Cauchy in $L^1(\mu)$. By (a), f_n is Cauchy in measure. By previous Prop, f_n is convergent in measure.

Now, we show (a). Fix $\varepsilon > 0$. Define

$$E_{n,m,\varepsilon} := \{x : |f_n(x) - f_m(x)| \geq \varepsilon\}.$$

Then,

$$\varepsilon \mu(E_{n,m,\varepsilon}) = \int_{E_{n,m,\varepsilon}} \varepsilon \, d\mu$$

$$\leq \int_{E_{n,m,\varepsilon}} |f_n(x) - f_m(x)| \, d\mu(x)$$

$$= \|f_n - f_m\|_{L^1(\mu)}.$$

We see that RHS $\to 0$ as $n, m \to \infty$.
Thus, LHS $\to 0$, hence f_n is Cauchy in measure.

MAJOR THEOREM 8

Cor: If f_n is Cauchy in $L^1(\mu)$, then $\exists f \in L^1(\mu)$ and a subsequence f_{n_k} s.t. $f_{n_k} \to f$ μ-a.e.

Recall: Any Cauchy sequence in a metric space is bounded.

Pf: By Prop, f_n is Cauchy in measure. By prev Thm, $\exists$ f measurable and subsequence f_{n_k} s.t. $f_{n_k} \to f$ μ-a.e.

It remains to show $f \in L^1(\mu)$.

By Fatou's Lemma,

$$\int |f| d\mu = \int \liminf_{k\to\infty} |f_{n_k}| d\mu \le \liminf_{k\to\infty} \|f_{n_k}\|_{L^1(\mu)} < +\infty,$$

since f_n is bounded in $L^1(\mu)$. $\square$

MAJOR THEOREM a

Cor: $L^1(\mu)$ is a Banach space, that is, a complete normed vector space.

Pf: Let f_n be Cauchy in $L^1(\mu)$.

By Cor, $\exists f \in L^1(\mu)$ and a subseq f_{n_k} s.t. $f_{n_k} \to f$ μ-a.e.

By Fatou's lemma,

$$\int |f_{n_k} - f| \, d\mu = \int \liminf_{j \to \infty} |f_{n_k} - f_{n_j}| \, d\mu$$

$$\leq \liminf_{j \to \infty} \int |f_{n_k} - f_{n_j}| \, d\mu$$

Since f_n is Cauchy in $L^1(\mu)$. RHS $\to 0$ as $k \to +\infty$. Thus $f_{n_k} \to f$ in $L^1(\mu)$.

Since $L^1(\mu)$ is a metric space, by Δ ineq, we must have $f_n \to f$ in $L^1(\mu)$. $\square$

Summary of Modes of Conv.

$f_n \to f$ in $L^1(\mu)$

$\Downarrow$ $\quad \not\Uparrow \ \frac{1}{n} 1_{[0,n]}$

$f_n \to f$ in measure

$\Downarrow$ up to a subsequence $\quad \Uparrow$?

$f_{n_k} \to f$ μ-a.e.

Next time: Egoroff's Theorem