# Lecture 13

Recall:

**Def:** A sequence of measurable functions $f_n : X \to \mathbb{R}$ converges in measure to a measurable function $f : X \to \mathbb{R}$ if, $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

Likewise, $f_n$ is Cauchy in measure if, $\forall \varepsilon > 0$,

$$\lim_{m,n\to\infty} \mu(\{x : |f_n(x) - f_m(x)| \geq \varepsilon\}) = 0.$$

**Thm:** Consider $f_n: X \to \mathbb{R}$ meas.

(i) If $f_n$ is Cauchy in measure, then $\exists\ f: X \to \mathbb{R}$ meas. s.t.

(a) $f_n \to f$ in measure

(b) $\exists\ f_{n_k}$ s.t. $f_{n_k} \to f$ $\mu$-a.e.

(ii) If, in addition, $f_n \to g$ in measure, then $f = g$ $\mu$-a.e.

**Prop:**

(a) If $f_n$ is Cauchy in $L^1(\mu)$, then it's Cauchy in measure.

(b) If $f_n$ is convergent in $L^1(\mu)$, then it's convergent in measure.

MAJOR THEOREM 8

Cor: If $f_n$ is Cauchy in $L^1(\mu)$, then $\exists\, f \in L^1(\mu)$ and a subsequence $f_{n_k}$ s.t. $f_{n_k} \to f$ $\mu$-a.e.

MAJOR THEOREM 9

Cor: $L^1(\mu)$ is a Banach space, that is, a complete normed vector space.

Summary of different modes of convergence:

$f_n \to f$ in $L^1(\mu)$

$\Downarrow$ $\quad\quad \not\Uparrow \ \frac{1}{n} 1_{[0,n]}$

$f_n \to f$ in measure $\quad \Uparrow$ ?

$\Downarrow$ upto a subsequence

$f_n \to f$ $\mu$-a.e.

To answer ?, we first show...

MAJOR THEOREM 10

Thm (Egoroff): Suppose $\mu(X) < +\infty$ and $f_n, f : X \to \mathbb{R}$ measurable s.t. $f_n \to f$ $\mu$-a.e..

Then, $\forall \varepsilon > 0$, $\exists$ $E \in \mathcal{M}$ s.t. $\mu(E) < \varepsilon$ s.t. $f_n \to f$ uniformly on $E^c$.

Pf:

$\boxed{\text{Case 1}}$ Assume $f_n \to f$ pointwise.

Define $E_{n,k} = \bigcup_{m=n}^{\infty} \{x : |f_m(x) - f(x)| \geq \frac{1}{k}\}$

Then $E_{n,k} \supseteq E_{n+1,k}$ $\forall n, k$.

Since $\mu(X) < +\infty$, cty from above ensures, for any $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \mu(E_{n,k}) = \mu\left(\bigcap_{n=1}^{\infty} E_{n,k}\right) = \mu(\emptyset) = 0.$$

$f_n \to f$ pointwise

$\forall x \in X, \forall k \in \mathbb{N}, \exists M > 0$ s.t. $\forall m \geq M$ $|f_m(x) - f(x)| < 1/k$

Fix $\varepsilon > 0$ arbitrary. Then $\forall k \in \mathbb{N}$, $\exists n_k \in \mathbb{N}$ s.t.

$$\mu(E_{n_k,k}) < \frac{\varepsilon}{2^k}.$$

Let $E = \bigcup_{k=1}^{\infty} E_{n_k,k}$. Then

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_{n_k,k}) \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

If $x \notin E$, then $x \notin E_{n_k, k}$ $\forall k \in \mathbb{N}$.
that is, for any $k \in \mathbb{N}$, $\forall m \geq n_k$,
$x \notin \{x : |f_m(x) - f(x)| \geq \frac{1}{k}\}$.

Therefore, $\forall k \in \mathbb{N}$,
$|f_m(x) - f(x)| < \frac{1}{k}$ $\forall m \geq n_k$ and $x \in E^c$.

Hence $f_n \to f$ unif on $E^c$.

$\boxed{\text{Case 2}}$ Assume $f_n \to f$ $\mu$-a.e.

Define $N := \{x : f_n(x) \not\to f(x)\}$,
so $\mu(N) = 0$.

Define $g_n := f_n 1_{N^c}$, $g := f 1_{N^c}$

Then $g_n \to g$ pointwise.

By Case 1, $\forall\, \varepsilon>0$, $\exists\, E \in \mathcal{M}$ with $\mu(E)<\varepsilon$ s.t. $g_n \to g$ unif on $E^c$.

Let $F := E \cup N$. Then $\mu(F) < \varepsilon + 0 = \varepsilon$. and $f_n = g_n \to g = f$ uniformly on $F^c$. $\square$

<u>Cor</u>: Suppose $\mu(X) < +\infty$ and $f_n, f : X \to \mathbb{R}$ measurable s.t. $f_n \to f$ $\mu$-a.e.. Then $f_n \to f$ in measure.

Pf: Fix $\varepsilon > 0$. By Egoroff, $\forall\, \delta>0$, $\exists\, E \in \mathcal{M}$ s.t. $\mu(E) < \delta$ s.t. $f_n \to f$ unif on $E^c$.

Thus,

$$\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\})$$

$$\leq \mu(\{x \in E^c : |f_n(x) - f(x)| \geq \varepsilon\}) + \underbrace{\mu(E)}_{<\delta}$$

Thus,

$$\limsup_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\})$$

$$\leq \underbrace{\limsup_{n\to\infty} \mu(\{x \in E^c : |f_n(x) - f(x)| \geq \varepsilon\})}_{=0} + \delta$$

Since $f_n \to f$ unif on $E^c$...

Sending $\delta \to 0$, we conclude $f_n \to f$ in measure. $\square$

# Product Measures

$X$ is a nonempty set

<u>Prop</u>: Given $\mathcal{E} \subseteq 2^X$, $\exists$ a smallest $\sigma$-algebra containing $\mathcal{E}$, denoted $\mathcal{M}(\mathcal{E})$.

<u>Rmk</u>

(a) if $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$

(b) if $\mathcal{E} \subseteq \mathcal{F}$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$

Let

- $\{(X_\alpha, \mathcal{M}_\alpha)\}_{\alpha \in A}$ be a countable collection of measurable spaces
- $X := \prod_{\alpha \in A} X_\alpha = X_1 \times X_2 \times \ldots \times X_\alpha \times \ldots$
- Let $\pi_\alpha$ denote projection of $X$ onto $X_\alpha$

**Def**: The *product $\sigma$-algebra* is

$$\bigotimes_{\alpha\in A} \mathcal{M}_\alpha := \mathcal{M}\Big(\Big\{\underbrace{\prod_{\alpha\in A} E_\alpha}_{\text{"rectangle"}} : E_\alpha \in \mathcal{M}_\alpha\Big\}\Big)$$

Ex: $\mathcal{M}_\alpha := \mathcal{B}_{\mathbb{R}}$, $E_\alpha := [a,b]$, $A = \{1,2\}$

**Goal**: WTS $\mathcal{B}_{\mathbb{R}^d} = \bigotimes_{\alpha=1}^{d} \mathcal{B}_{\mathbb{R}}$

**Prop**: Given $\mathcal{E}_\alpha \subseteq 2^{X_\alpha}$ s.t. $X_\alpha \in \mathcal{E}_\alpha$, suppose $\mathcal{M}_\alpha = \mathcal{M}(\mathcal{E}_\alpha)$. Then

$$\bigotimes_{\alpha\in A} \mathcal{M}_\alpha = \mathcal{M}\Big(\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\Big)$$

Ex: $\mathcal{E}_\alpha = \{(a,+\infty) : a\in\mathbb{R} \text{ or } a=-\infty\}$

Pf: By (b),

$$\mathcal{M}\left(\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\right)$$

$$\subseteq \mathcal{M}\left(\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\right\}\right).$$

We now show the opposite containment. By (a), it suffices to show

$$\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\right\}$$

$$\subseteq \mathcal{M}\left(\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\right).$$

Note that...

$$\prod_{\alpha\in A} E_\alpha = \{x \in X : \pi_\alpha(x) \in E_\alpha \ \forall \alpha \in A\}$$
$$= \bigcap_{\alpha\in A} \{x \in X : \pi_\alpha(x) \in E_\alpha\}$$

$$\dots = \underbrace{\bigcap_{\alpha \in A}}_{} \pi_\alpha^{-1}(E_\alpha)$$

(*) countable intersection

Thus, it suffices to show,
$\forall\, \alpha \in A$ and $E_\alpha \in \mathcal{M}_\alpha$,

$$\pi_\alpha^{-1}(E_\alpha) \in \overbrace{\mathcal{M}\left(\left\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\right\}\right)}^{\mathcal{A} :=}$$

Recall that, since $\mathcal{A}$ is a $\sigma$-algebra, its pushforward under the function $\pi_\alpha : X \to X_\alpha$ is also a $\sigma$-algebra, where...

$$\mathcal{F}_\alpha := \{ E \subseteq X_\alpha : \pi_\alpha^{-1}(E) \in \mathcal{A} \}$$

Furthermore, because $X_\alpha \in \mathcal{E}_\alpha$ $\forall\, \alpha \in A$ and

$$\pi_\alpha^{-1}(E_\alpha) = X_1 \times X_2 \times \dots \times E_\alpha \times X_{\alpha+1} \times \dots,$$

we have, $\forall\, E_\alpha \in \mathcal{E}_\alpha$,

$$\pi_\alpha^{-1}(E_\alpha) \in \left\{ \prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \right\} \subseteq \mathcal{A}.$$

Thus, $\mathcal{E}_\alpha \subseteq \mathcal{F}_\alpha$. Hence $\mathcal{M}_\alpha \subseteq \mathcal{F}_\alpha$.

This shows $(*)$. $\square$

topological: see Bogachev 6.4.2

Thm: Given $\overbrace{\text{metric spaces}}$ $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ and

$$\mathcal{X} := \prod_{i=1}^{n} \mathcal{X}_i$$

endowed with the metric

$$d_{\max}(\overbrace{(x_1, x_2, \ldots, x_n)}^{\vec{x}}, \overbrace{(y_1, y_2, \ldots, y_n)}^{\vec{y}}) = \max_{i=1,\ldots,n} d_i(x_i, y_i).$$

Then, $\bigotimes_{i=1}^{n} \mathcal{B}_{\mathcal{X}_i} \subseteq \mathcal{B}_{\mathcal{X}}$.

Furthermore, if the $\mathcal{X}_i$'s are all $\underbrace{\text{separable}}_{\exists \text{ countable dense subset.}}$, then $\mathcal{B}_{\mathcal{X}} = \bigotimes_{i=1}^{n} \mathcal{B}_{\mathcal{X}_i}$.

Rmk: $d_{max}$ is convenient because

$$B_r(x_1,\dots,x_n) = \{(y_1,\dots,y_n) : d_{max}(\vec{x},\vec{y}) < r\}$$
$$= \{\vec{y} : d_i(x_i,y_i) < r \ \forall i\}$$
$$= \prod_{i=1}^{n} B_r(x_i)$$

Rmk: However, since the defn of $\mathcal{B}_X$ only depends on the topology of $X$, this result continues to hold if $d_{max}$ is replaced by any equivalent metric.