Lecture 13 Recall :

Del: Asequence of measurable functions $f_n: X \rightarrow \mathbb{R}$ converges in measure to a measurable function $f: X \rightarrow \mathbb{R}$; f, $Y \in 20$,

 $\lim_{n \to \infty} \mu[\{x: | f_n(x) - f(x)| \ge \xi \} = 0.$

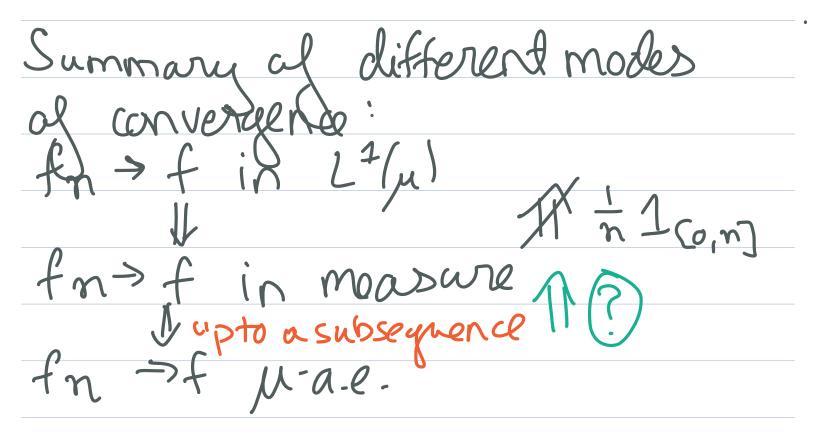
Likewise, fn is Cauchy in measure if, Y E>0,

 $\lim_{m,n\to\infty} \mu[\{x:|f_n(x)-f(x)] \ge \xi \} = 0.$

"Ihm Consider frix -> R meas. (i) I f fn is Cauchy in measure, then I f: X-> R meas. s.t. (a) fn→f in measure (i) If, in addition, fn=2q in measure, then f=& era.e. Brop: (a) If fn is (auchy in L¹(u), then it's (auchy is measure. (b) If fn is convergent in L¹(u), then it's convergent in measure.

Cor: If fn is Cauchy in L'(u) then J f E L I (u) and a subsequence fnx S.t. fnx→f vra.e.

MAJOR THEOREM a Cor: L²(u) is a Banach space, that is, a complete normal rector space.



To answer (), we first show... MAJOR THEOREM 10 Thru (Egorolf): Suppose $\mu(x) < too$ $and fm f: <math>\mathcal{K} \supset \mathcal{R}$ reasonable s.t. fn \supset f $\mu a e$.

Then, $\forall \epsilon > 0$, $\exists E \epsilon \mathcal{M} s.t.$ $\mu(\epsilon) < \epsilon s.t.$ $f_n \Rightarrow f uniformly$ $on E^{c}$

Pf: Case 1] Assume fn >f pointwise. Define $E_{n,k} = \bigcup_{m=n}^{\infty} \chi \cdot [f_m(x) - f(x)] = \frac{1}{k!}$ Then En, KZ Entr, K Yn, K.

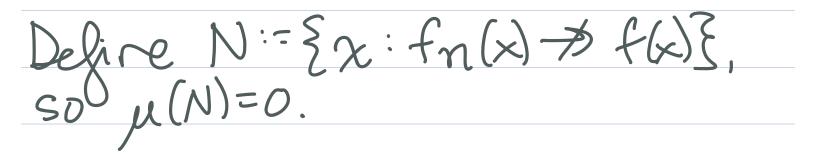
Since u(x)<to, cty from above ensures, for any KEN, $\lim_{n \to \infty} \mu(E_{n,k}) = \mu(\bigcap_{n=1}^{\infty} E_{n,k}) f_n \Rightarrow f_{n=1}$ $= \mu(\emptyset) = 0.$ $\neq x \in X, \forall k \in \mathbb{N}, \exists M = 0 s.t. \forall m z M$ Fix E>U ar britary. Then ¥KEIN, En KEIN s.t. MEnKK) < ZK. Let E= U Enk, K. Then K=1 Enk, K. Then $\mu(E) \in \underset{k=1}{\overset{\infty}{\rightarrow}} \mu(E_{n_{k}}, k) \leq \underset{k=1}{\overset{\infty}{\rightarrow}} \frac{\mathcal{E}}{2^{k}} = \mathcal{E}.$

If x & E, then x & Ener & V K & M. that is, for any k & M, Ym = nk, X & X: If m(x) - f(x) = 1/2 .

Thereferre, YKEIN, IF m(2) - f(x) < F Ym=nkand xEEC.

Hence fn >f unifon E?

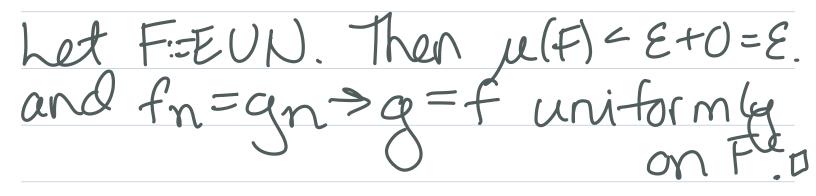
Case 2 Assume fn > f pra.e.



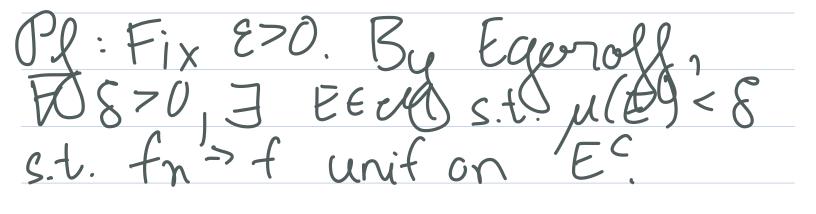
Define gni=fn 1_{Nc}, g:=f1_{Nc}



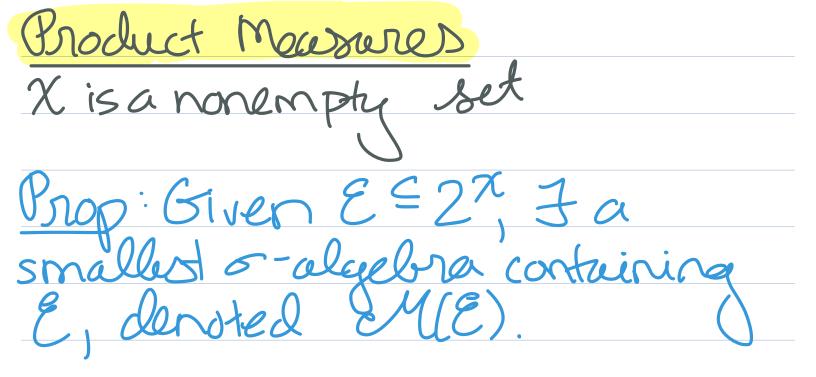
By Case 1, $\forall \epsilon^{20}$, $\exists \epsilon \epsilon cm$ with $\mu(\epsilon) < \epsilon s.t. q_n = q$ unifor ϵ^{c} .



Cor: Suppose $\mu(x) < +\infty$ and $f_n, f: x > \mathbb{R}$ measurable s.t. $f_n > f_{\mu-a.e.}$ Then $f_n > f_{in}$ measure.



Thus, $\mu[\{x: |f_n(x) - f(x)| \ge \varepsilon_{1}^{2}\})$ $=\mu \{x \in E^{c}: |f_{n}(x) - f(x)| \ge \sum_{i=1}^{n} \{y \in E^{c}\} \}$ Thub, $\lim_{n\to\infty} \mu[\{\chi: |f_n(\chi) - f(\chi)| \ge \xi]$) $\leq \limsup_{n \to \infty} \sup_{x \in \mathcal{X} \in \mathcal{E}} (f_n(x) - f(x)) \geq \varepsilon_1^2 + \delta$ Since fn->f unif on E^c... Sending $S \rightarrow 0$, we conclude $f_n \rightarrow f$ in measure. \Box

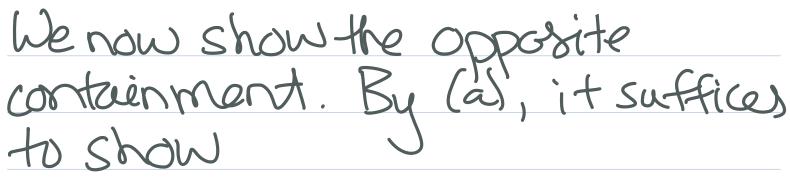


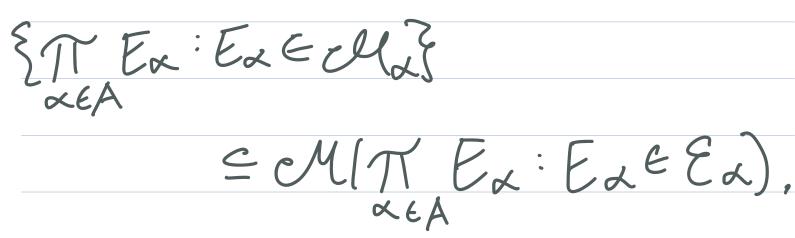
Rmk Kmk (a) if $E \subseteq \mathcal{M}(\mathcal{F}_{1})$, then $\mathcal{M}(E) \subseteq \mathcal{M}(\mathcal{F})$ (b) if $E \subseteq \mathcal{F}_{1}$, then $\mathcal{M}(E) \subseteq \mathcal{M}(\mathcal{F})$

 Let
E(Xa, Ma)Jacq be a countable collection of measurable conces spaces • $\chi^{T} := \pi \chi_{\alpha} = \chi_{1} \times \chi_{2} \times \ldots \times \chi_{\alpha} \times \ldots$ aeA · Let The clenote projection of Konto Xa

Def: The product 5-algebra is & Mz := METEx : Eze Mz zea "rectange" $E_{X}: \mathcal{M}_{x}:= \mathbb{B}_{R}, E_{x}:= [a,b], A= \{1,2\}$ Goal: WTS $B_{Rd} = \bigotimes B_{R}$ Prop: Given $\mathcal{E}_{\mathcal{A}} \subseteq 2^{\mathcal{X}_{\mathcal{A}}} \text{ s.t. } \mathcal{X}_{\mathcal{A}} \in \mathcal{E}_{\mathcal{A}},$ suppose $\mathcal{M}_{\mathcal{A}} = \mathcal{M}(\mathcal{E}_{\mathcal{A}})$. Then $(X) \mathcal{M}_{\alpha} = \mathcal{M}(\mathcal{T} \mathcal{F}_{\alpha} : \mathcal{F}_{\alpha} \in \mathcal{F}_{\alpha})$ $\alpha \in \mathcal{A}$ $\{\chi: \mathcal{E}_{\mathcal{A}} = \{(a, +\infty): a \in \mathbb{R} \text{ or } a = -\infty\}\}$

Pf: By (6), $M[\pi E_{\mathcal{X}}: E_{\mathcal{X}}\in \mathcal{E}_{\mathcal{A}})$ E METTER: ExEMD.





Note that ... = $\bigcap_{x \in X} (x) \in \mathbb{Z}^{2}$

 $\dots = \bigcap \pi_{\mathcal{L}}^{-1}(E_{\mathcal{L}})$ $\mathcal{L}eA$ Countable intersection Thus, it suffices to show, $\forall a \in A$ and $\exists a \in \mathcal{M}_{\mathcal{A}}$, A :=MJ'(EL) E MISTEL : ELEED) «EA Recall that, since chis a 5-algebra, its pushforward under the function Mr.X-7X2 is also a 5-algebra, where. $\mathcal{F}_{1} := \{ E \subseteq \chi_{\alpha} : \pi_{1}^{-1} | E \} \in \mathcal{C}_{\lambda} \}$

Furthermore, because X2EEx VXEA and

 $\pi_{\mathcal{L}}^{-1}(E_{\mathcal{L}}) = \chi_1 \times \chi_2 \times \ldots \times E_{\mathcal{L}} \times \chi_{\mathcal{A}^{+1}}$



TIZ (EX)ETTEX: EZEEZE.

Thus, $\mathcal{E}_{a} \subseteq \mathcal{F}_{a}$. Hence $\mathcal{M}_{a} \subseteq \mathcal{F}_{a}$.



topological: see Bogacher 6.4.2 Thm: Given metricspaces $\chi_1, \chi_2, ..., \chi_n$ and $\mathcal{X} := \mathcal{T} \mathcal{X}_{\mathcal{U}}^{\cdot}$ endowed with the metric $d_{max}([x_{1}, x_{2}, ..., x_{n}], (y_{1}, y_{2}, ..., y_{n}])$ = max $d_{i}(x_{i}, y_{i})$.

Then, $\bigotimes B_{\chi} \subseteq B_{\chi}$.

Furthermore, if the χ_i 's are all se parable, then $B\chi = \bigotimes B_{\chi_i}$. \exists constable dense subset.

Kmk: Imax is convenient because

 $B_{r}(x_{i}, x_{n}) = \frac{1}{2} (y_{i}, y_{n}) \cdot d_{max}(\vec{x}, \vec{y}) < r$ $= \frac{1}{2} \cdot \frac{1}{2} \cdot$

Kmk: However, since the defn of Bx only depends on the topology of X, this result continues to hold if dmax is replaced by any equivalent matric.