Lecture 14

Recall:

MAJOR THEOREM 10

Thm (Egoroff): Suppose $\mu(X) < +\infty$ and $f_n, f : X \to \mathbb{R}$ measurable s.t. $f_n \to f$ μ-a.e.

Then, $\forall \varepsilon > 0$, $\exists$ $E \in \mathcal{M}$ s.t. $\mu(E) < \varepsilon$ s.t. $f_n \to f$ uniformly on E^c.

Cor: Suppose $\mu(X) < +\infty$ and $f_n, f : X \to \mathbb{R}$ measurable s.t. $f_n \to f$ μ-a.e. Then $f_n \to f$ in measure.

Product Measures

X is a nonempty set

Prop: Given $\mathcal{E} \subseteq 2^X$, $\exists$ a smallest σ-algebra containing $\mathcal{E}$, denoted $\mathcal{M}(\mathcal{E})$.

Let

- $\{(X_\alpha, \mathcal{M}_\alpha)\}_{\alpha \in A}$ be a countable collection of measurable spaces
- $X := \prod_{\alpha \in A} X_\alpha = X_1 \times X_2 \times \dots \times X_\alpha \times \dots$
- Let π_α denote projection of X onto X_α

Def: The product σ-algebra is

$$\bigotimes_{\alpha \in A} \mathcal{M}_\alpha := \mathcal{M}\Big(\Big\{\underbrace{\prod_{\alpha\in A} E_\alpha}_{\text{"rectangle"}} : E_\alpha \in \mathcal{M}_\alpha\Big\}\Big)$$

Prop: Given $\mathcal{E}_\alpha \subseteq 2^{X_\alpha}$ s.t. $X_\alpha \in \mathcal{E}_\alpha$, suppose $\mathcal{M}_\alpha = \mathcal{M}(\mathcal{E}_\alpha)$. Then

$$\bigotimes_{\alpha \in A} \mathcal{M}_\alpha = \mathcal{M}\Big(\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\Big)$$

topological: see Bogachev 6.4.2

Thm: Given $\overbrace{\text{metric spaces}}$ $X_1, X_2, \dots, X_n$ and

$$X := \prod_{i=1}^{n} X_i$$

endowed with the metric

$$d_{max}(\overbrace{(x_1, x_2, \dots, x_n)}^{\vec{x}}, \overbrace{(y_1, y_2, \dots, y_n)}^{\vec{y}}) = \max_{i=1,\dots,n} d_i(x_i, y_i).$$

Then, $\bigotimes_{i=1}^{n} \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$.

Furthermore, if the X_i's are all separable, then $\mathcal{B}_X = \bigotimes_{i=1}^{n} \mathcal{B}_{X_i}$.

$\exists$ countable dense subset.

Rmk: d_{max} is convenient because

$$B_r(x_1,\dots,x_n) = \{(y_1,\dots,y_n) : d_{max}(\vec{x},\vec{y}) < r\}$$
$$= \{\vec{y} : d_i(x_i,y_i) < r \ \forall i\}$$
$$= \prod_{i=1}^{n} B_r(x_i)$$

Rmk: However, since the defn of $\mathcal{B}_X$ only depends on the topology of X, this result continues to hold if d_{max} is replaced by any equivalent metric.

Recall:

Fact #1: If $X_1, X_2, \dots, X_n$ are n separable, then so is $X = \prod_{i=1} X_i$.

Fact #2: In a separable metric space, every open set can be written as a countable union of (open) balls.

Fact #3:

$$\left\{\prod_{i=1}^{n} E_i : E_i \subseteq X_i, E_i \text{ open}\right\} \subseteq \{\mathcal{U} : \mathcal{U} \subseteq X, \mathcal{U} \text{ open}\}$$

<u>Pf</u>: For the first part of the theorem, note that, by Prop,

$$\bigotimes_{i=1}^{n} \mathcal{B}_{X_i} = \mathcal{M}\left(\left\{\prod_{i=1}^{n} E_i : E_i \subseteq X_i, E_i \text{ open}\right\}\right)$$

$$\overset{\text{Fact \#3}}{\subseteq} \mathcal{M}(\{\mathcal{U} : \mathcal{U} \subseteq X, \mathcal{U} \text{ open}\})$$

$$=: \mathcal{B}_X.$$

Now, suppose $X_1, X_2, \ldots, X_n$ are separable. By Facts #1, 2, every open subset of X can be written as a countable union of balls. Thus, it suffices to show that, for any ball $B \subseteq X$,

$$B \in \bigotimes_{i=1}^{n} \mathcal{B}_{X_i} = \mathcal{M}\left(\left\{\prod_{i=1}^{n} E_i : E_i \subseteq X_i,\ E_i \text{ open}\right\}\right)$$

Since X is endowed with d_{max}, $B = \prod_{i=1}^{n} B_i$, for $B_i \subseteq X_i$, B_i open ball, this is immediate. $\square$

Product Measures

Measure spaces $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$

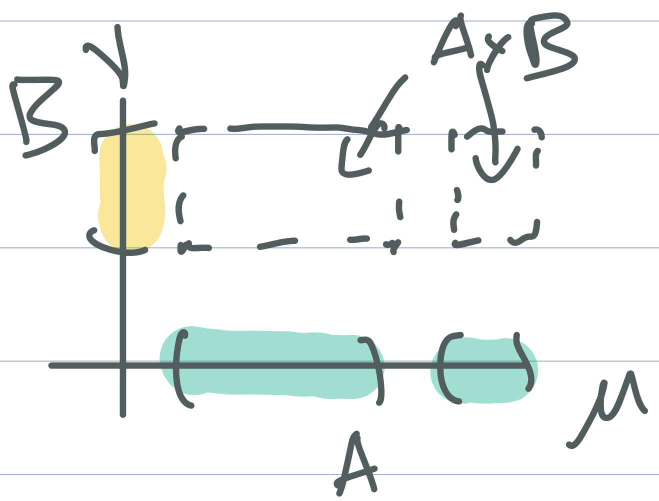
Rectangles $A \times B$, $A \in \mathcal{M}$, $B \in \mathcal{N}$

$$\mathcal{M} \otimes \mathcal{N} := \sigma(\{A \times B : A \in \mathcal{M}, B \in \mathcal{N}\})$$

Goal: Prove existence of a unique measure $\mu \otimes \nu$ on the measurable space $(X \times Y, \mathcal{M} \otimes \mathcal{N})$ so that

$$\mu \otimes \nu (A \times B) = \mu(A) \nu(B)$$

for all rectangles with $\mu(A), \nu(B) < +\infty$.



This motivates the development of necessary conditions to ensure a measure is unique...

... will do with Monotone Class Theorem.

<u>Recall:</u>

Fix a nonempty subset X.

<u>Def:</u> $\mathcal{A}$ is an algebra of subsets of X if it is a nonempty collection that is closed under finite unions and complements.

<u>Rmk:</u> $\emptyset, X \in \mathcal{A}$

Def: $C \subseteq 2^X$ is a <u>monotone class</u> iff it is a nonempty collection s.t.

(i) $\{E_i\}_{i=1}^{\infty} \subseteq C$, $E_1 \subseteq E_2 \subseteq \ldots \Rightarrow \bigcup_{i=1}^{\infty} E_i \in C$

"closed under countable increasing unions"

(ii) $\{E_i\}_{i=1}^{\infty} \subseteq C$, $E_1 \supseteq E_2 \supseteq \ldots \Rightarrow \bigcap_{i=1}^{\infty} E_i \in C$

"closed under countable decreasing intersections"

Ex: Any σ-algebra is a monotone class.

Ex: $X = \{1, 2, 3\}$

$C = \{\{1\}, \{1,2\}\}$ is a monotone class, but not a σ-algebra.

Prop: Given any $\mathcal{E} \subseteq 2^X$ nonempty, there exists a smallest monotone class containing $\mathcal{E}$, denoted $\mathcal{C}(\mathcal{E})$.

Pf: HW or practice midterm

Thm: (Monotone Class Theorem): Given an algebra $\mathcal{A} \subseteq 2^X$,

$$\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$$

Pf: Since $\mathcal{M}(\mathcal{A})$ is a monotone class, we immediately obtain $\mathcal{C}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$.

To prove $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{C}(\mathcal{A})$, it suffices to show $\underbrace{\mathcal{C}(\mathcal{A})}_{=:\mathcal{C}}$ is a σ-algebra.

For any $E \in \mathcal{C}$, define

$$\mathcal{E}_E = \{F \in \mathcal{C} : E \setminus F, F \setminus E, F \cap E \in \mathcal{C}\}$$

(i) Note that $E, \emptyset \in \mathcal{E}_E$

(ii) $F \in \mathcal{E}_E \Leftrightarrow E \in \mathcal{E}_F$

(iii) if $E, F \in \mathcal{A}$, then $F \in \mathcal{E}_E$.

(iv) CLAIM: For any $E \in \mathcal{C}$, $\mathcal{E}_E$ is a monotone class

Pf of Claim:

If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{E}_E$, $E_1 \subseteq E_2 \subseteq \dots$

WTS $\bigcup_{i=1}^{\infty} E_i \in \mathcal{E}_E$.

- $E \setminus \left(\bigcup_{i=1}^{\infty} E_i\right) = E \cap \left(\bigcap_{i=1}^{\infty} E_i^c\right)$

$$= \bigcap_{i=1}^{\infty} (E \cap E_i^c) = \underbrace{\bigcap_{i=1}^{\infty} (\overbrace{E \setminus E_i}^{\in \mathcal{C} \text{ for each } i})}_{}$$

decreasing intersection $\in \mathcal{C}$

- $\left(\bigcup_{i=1}^{\infty} E_i\right) \setminus E = \bigcup_{i=1}^{\infty} (E_i \setminus E) \in \mathcal{C}$

- $\left(\bigcup_{i=1}^{\infty} E_i\right) \cap E = \bigcup_{i=1}^{\infty} (E_i \cap E) \in \mathcal{C}$

Thus $\bigcup_{i=1}^{\infty} E_i \in \mathcal{E}_E$

Via a similar argument, $\mathcal{E}_E$ is closed under decreasing intersections, so $\mathcal{E}_E$ is a monotone class.

Combining (iii) and (iv), for any $E \in \mathcal{A}$, $\mathcal{A} \subseteq \mathcal{E}_E$ and since $\mathcal{E}_E$ is a monotone class, $\mathcal{C} \subseteq \mathcal{E}_E$, that is...

$\forall E \in \mathcal{A}, F \in \mathcal{C},$

$$F \in \mathcal{E}_E \overset{?}{\Longleftrightarrow} E \in \mathcal{E}_F.$$

Thus, $\forall F \in \mathcal{C}$, $\mathcal{A} \subseteq \mathcal{E}_F$, thus, $\mathcal{C} \subseteq \mathcal{E}_F$.

Since $X \in \mathcal{C}$, $X \in \mathcal{E}_F$, so $X \setminus F = F^c \in \mathcal{C}$.

This shows $\mathcal{C}$ is closed under complements.

Likewise, for any $E, F \in \mathcal{C}$, $E \cap F \in \mathcal{C}$.

Furthermore, if $E_1, E_2 \in \mathcal{C}$,

$$E_1 \cup E_2 = \underbrace{(\underbrace{E_1^c}_{\in \mathcal{C}} \cap \underbrace{E_2^c}_{\in \mathcal{C}})}_{\in \mathcal{C}}{}^c \in \mathcal{C}.$$

Thus, $\mathcal{C}$ is closed under finite unions. Hence $\mathcal{C}$ is an algebra.

On HW2, you showed that if an algebra is closed under countable increasing unions, it is a

σ-algebra. Therefore $\mathcal{C}$ is a σ-algebra. $\square$

Now: sufficient conditions for uniqueness of measures.

Thm (uniqueness of measures)

Suppose...

- $\mathcal{A} \subseteq 2^X$ is an algebra
- μ and ν are measures on $\mathcal{M}(\mathcal{A})$.
- $\exists \{A_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ s.t. $\bigcup_{i=1}^{\infty} A_i = X$, $\mu(A_i) < +\infty$, $\forall i$.

"strong σ-finiteness"

Then $\mu(A) = \nu(A)$ $\forall A \in \mathcal{A} \Rightarrow \mu = \nu$.

Recall from lecture:
Given measure space $(X, \mathcal{M}, \mu)$, $\forall A \in \mathcal{M}$,

$$\mu_A(E) := \mu(E \cap A)$$

is a measure on $(X, \mathcal{M})$.

Pf:

Case 1: $\mu(X) < +\infty$ (so $\nu(X) < +\infty$).

Consider $\mathcal{E} = \{E \in \mathcal{M}(\mathcal{A}) : \mu(E) = \nu(E)\}$
Since $\mathcal{A} \subseteq \mathcal{E}$, it suffices to show $\mathcal{E}$ is a monotone class to conclude

$$\mathcal{M}(\mathcal{A}) = \mathcal{C}(\mathcal{A}) \subseteq \mathcal{E}.$$

• Fix $\{B_i\}_{i=1}^{\infty} \in \mathcal{E}$, $B_1 \subseteq B_2 \subseteq \ldots$.
Then, by cty from below

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{i\to\infty} \mu(B_i)$$

$$= \lim_{i\to\infty} \nu(B_i) = \nu\left(\bigcup_{i=1}^{\infty} E_i\right)$$

Thus $\bigcup_{i=1}^{\infty} B_i \in \mathcal{E}$.

• Similarly, $\mathcal{E}$ is closed under countable decreasing intersections by cty from above, since $\mu(X) < +\infty$.

Thus $\mathcal{E}$ is a monotone class.

<u>Case 2</u>: $\mu(X) = +\infty$

WLOG, we may assume A_i in strong σ-finiteness hypothesis are disjoint.

(Otherwise, $B_1 = A_1$, $B_2 = A_2 \setminus A_1, \ldots$)

Define $\mu_i(E) = \mu(A_i \cap E)$
$\nu_i(E) = \mu(A_i \cap E)$

These are finite measures on X.
By case 1, $\mu_i = \nu_i$.

Finally, for any $E \in \sigma(\mathcal{A})$,

$\bigcup_{i=1}^{\infty} A_i = X$ $\quad$ A_i disjoint

$$\mu(E) = \mu\Big(\bigcup_{i=1}^{\infty} (E \cap A_i)\Big) = \sum_{i=1}^{\infty} \overbrace{\mu(E \cap A_i)}^{\mu_i(E)}$$

$$\ldots = \sum_{i=1}^{\infty} \underbrace{\nu(E \cap A_i)}^{\nu_i(E)} = \nu(E).$$

Thus $\mu = \nu$. $\square$

End of material for Midterm 2