Lecture 15

IMPORTANT MESSAGE ABOUT MIDTERM 2:

On the midterm, there is one question where part (b) says, "Use part (a) to show..." -> You really need to use part (a).

(Another solution to this problem would be to use a homework problem, but that is not what I am looking for.)

Recall:

Product Measures

Measure spaces $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$

Rectangles $A \times B$, $A \in \mathcal{M}$, $B \in \mathcal{N}$

$$\mathcal{M} \otimes \mathcal{N} := \mathcal{M}(\{A \times B : A \in \mathcal{M}, B \in \mathcal{N}\})$$

Goal: Prove existence of a unique measure $\mu \otimes \nu$ on the measurable space $(X \times Y, \mathcal{M} \otimes \mathcal{N})$ so that

$\mu \otimes \nu(A \times B) = \mu(A)\nu(B)$

for all rectangles with $\mu(A), \nu(B) < +\infty$.

Def: $C \subseteq 2^X$ is a monotone class iff it is a nonempty collection s.t.

(i) $\{E_i\}_{i=1}^{\infty} \subseteq C$, $E_1 \subseteq E_2 \subseteq \ldots \Rightarrow \bigcup_{i=1}^{\infty} E_i \in C$
"closed under countable increasing unions"

(ii) $\{E_i\}_{i=1}^{\infty} \subseteq C$, $E_1 \supseteq E_2 \supseteq \ldots \Rightarrow \bigcap_{i=1}^{\infty} E_i \in C$
"closed under countable decreasing intersections"

Prop: Given any $\mathcal{E} \subseteq 2^X$ nonempty, there exists a smallest monotone class containing $\mathcal{E}$, denoted $\mathcal{C}(\mathcal{E})$.

Thm: (Monotone Class Theorem): Given an algebra $\mathcal{A} \subseteq 2^X$,

$$\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$$

Thm (uniqueness of measures)

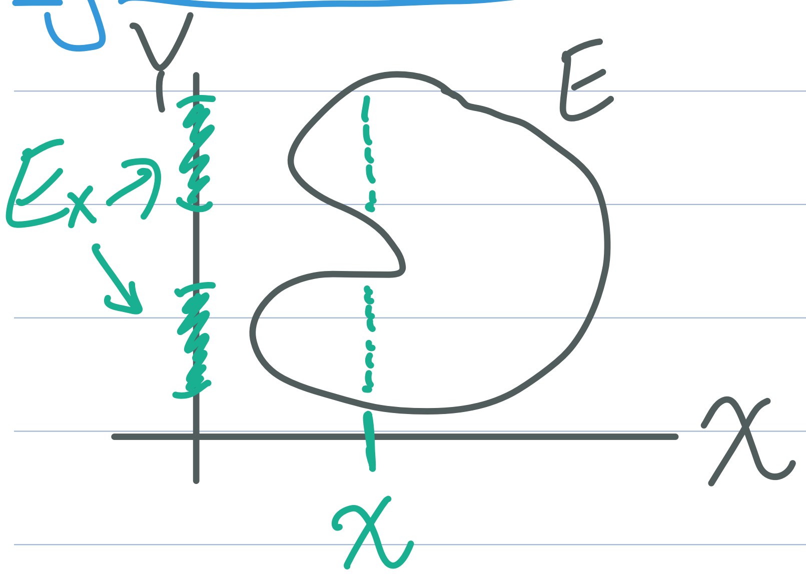
Suppose...

- $\mathcal{A} \subseteq 2^X$ is an algebra
- μ and ν are measures on $\mathcal{M}(\mathcal{A})$. "strong σ-finiteness"
- $\exists \{A_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ s.t. $\bigcup_{i=1}^{\infty} A_i = X$, $\mu(A_i) < +\infty$, $\forall i$.

Then $\mu(A) = \nu(A)$ $\forall A \in \mathcal{A} \Rightarrow \mu = \nu$.

We now have what we need to return to our goal of proving $\exists!\ \mu\otimes\nu$...

Def: For any $E\in\mathcal{M}\otimes\mathcal{N}$, the <u>x-section</u> is $E_x=\{y:(x,y)\in E\}$
<u>y-section</u> $E^y=\{x:(x,y)\in E\}$



<u>Basic properties of sections</u>:

(A) If $E = A \times B$, $A \in \mathcal{M}$, $B \in \mathcal{N}$

$$E_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A, \end{cases} \qquad E^y = \begin{cases} A & \text{if } y \in B \\ \emptyset & \text{if } y \notin B \end{cases}$$

(B) Given $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M} \otimes \mathcal{N}$,

$$\left(\bigcup_{i=1}^{\infty} E_i\right)_x = \left\{y : (x,y) \in \bigcup_{i=1}^{\infty} E_i\right\}$$

$$= \bigcup_{i=1}^{\infty} \{y : (x,y) \in E_i\} = \bigcup_{i=1}^{\infty} (E_i)_x$$

(C) Given $E \in \mathcal{M} \otimes \mathcal{N}$

$$(E^c)_x = \{y : (x,y) \notin E\}$$

$$= \{y : (x,y) \in E\}^c = (E_x)^c.$$

Prop: If $E \in \mathcal{M} \otimes \mathcal{N}$ then
(*) $[E_x \in \mathcal{N},\ E^y \in \mathcal{M}\ \forall x \in X, y \in Y.$

Pf: Let $\mathcal{E} = \{E \in \mathcal{M} \otimes \mathcal{N} : (*) \text{ holds}\}$

WTS $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{E}$.

By (A), $\mathcal{E}$ contains all rectangles.

Since $\mathcal{M} \otimes \mathcal{N}$ is generated by rectangles, it suffices to show $\mathcal{E}$ is a σ-algebra. This follows from properties (B) and (C). □

Basic Properties of Sections (Cont.)

(D) If $E \in \mathcal{M} \otimes \mathcal{N}$, then

$$\nu(E_x) = \int_Y 1_{E_x}(y)\, d\nu(y) = \int_Y 1_E(x,y)\, d\nu(y)$$

Thm: Consider σ-finite measure spaces $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$.

For any $E \in \mathcal{M} \otimes \mathcal{N}$,

(i) the functions
$x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$
are $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ and $(\mathcal{N}, \mathcal{B}_{\mathbb{R}})$-meas.

(ii) $\int_X \nu(E_x)\,d\mu(x) = \int_Y \mu(E^y)\,d\nu(y)$

Spoiler: (ii) is equivalent to showing

$$\int_X \int_Y 1_E(x,y)\,d\nu(y)\,d\mu(x) = \int_Y \int_X 1_E(x,y)\,d\mu(x)\,d\nu(y)$$

$$=: \mu\otimes\nu(E)$$

we will define $\mu\otimes\nu$ in this way

First, a lemma...

<u>Lemma</u>: Consider measurable spaces $(X,\mathcal{M}), (Y,\mathcal{N})$. Let $\mathcal{A} := \{\bigcup_{i=1}^n E_i : \{E_i\}_{i=1}^n$ are disjoint rectangles and $n\in\mathbb{N}\}$

Then $\mathcal{A}$ is an algebra and $\mathcal{M}(\mathcal{A}) = \mathcal{M} \otimes \mathcal{N}$.

Pf: For $\mathcal{A}$ is an algebra, see HW7.

Since all rectangles belong to $\mathcal{A}$, $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{M}(\mathcal{A})$.

Since $\mathcal{A} \subseteq \mathcal{M} \otimes \mathcal{N}$ and RHS is a σ-algebra, $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{M} \otimes \mathcal{N}$. $\square$

Pf of <u>Thm</u>:

$\boxed{\text{Case 1}}$ Suppose $\mu(X), \nu(Y) < +\infty$.

Let $\mathcal{C} = \{E \in \mathcal{M} \otimes \mathcal{N} : \text{(i) and (ii) hold}\}$

Our goal is to show $\mathcal{C} = \mathcal{M} \otimes \mathcal{N}$.

Strategy for proof:

(I) show $\mathcal{A} \subseteq \mathcal{C}$, for

$$\mathcal{A} := \left\{ \bigcup_{i=1}^{n} E_i : \{E_i\}_{i=1}^{n} \text{ are disjoint rectangles and } n \in \mathbb{N} \right\}$$

(II) show $\mathcal{C}$ is a monotone class

Then, $\mathcal{M} \otimes \mathcal{N} \overset{\text{Lemma}}{=} \mathcal{M}(\mathcal{A}) \overset{\text{Monotone Class Thm}}{=} \mathcal{C}(\mathcal{A}) \subseteq \mathcal{C}$

We begin with (I). Suppose $E \in \mathcal{A}$, that is $E = \bigcup_{i=1}^{n} E_i$, $E_i = A_i \times B_i$.

disjoint $\quad A_i \in \mathcal{M} \quad B_i \in \mathcal{N}$

By property (B),

$$E_x = \left(\bigcup_{i=1}^{n} E_i\right)_x = \bigcup_{i=1}^{n} E_{i,x}.$$ disjoint

Thus, $E_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$

$$\nu(E_x) = \sum_{i=1}^{n} \nu(E_{i,x}) = \sum_{i=1}^{n} 1_{A_i}(x)\nu(B_i)$$

This is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable.
Similarly, $\mu(E^y)$ is $(\mathcal{N}, \mathcal{B}_{\mathbb{R}})$-meas.
Thus E satisfies (i).

Furthermore,

$$\int_X \nu(E_x)\,d\mu(x) = \int_X \sum_{i=1}^{n} 1_{A_i}(x)\nu(B_i)\,d\mu(x)$$

$$= \sum_{i=1}^{n} \mu(A_i)\nu(B_i)$$

$$I_{\dots} = \int_Y \mu(E^y)\, d\nu(y).$$

Thus, E satisfies (ii).

Now, we prove (II), that is, $\mathcal{C}$ is a monotone class.

Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{C}$, $E_1 \subseteq E_2 \subseteq \dots$

WTS $E := \bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$.

Note that $(E_1)^y \subseteq (E_2)^y \subseteq \dots \quad \forall x \in X$

$(E_1)_x \subseteq (E_1)_x \subseteq \dots \quad y \in Y.$

Thus $f_n(y) := \mu(E_n^y)$ and $g_n(x) := \nu((E_n)_x)$ are nonneg, meas fns.

By cty from below, $\forall x \in X, y \in Y$,

$f_n(y) \nearrow \mu(E^y)$, $g_n(x) \nearrow \nu(E_x)$.

Thus E satisfies (i). Furthermore, by Monotone Convergence Thm,

$$\left.\begin{aligned}
\int_Y \mu(E^y)\,d\nu(y) &= \lim_{n\to\infty} \int_Y \overbrace{f_n(y)}^{=\mu(E_n^y)}\,d\nu(y) \\
&= \lim_{n\to\infty} \int_X \overbrace{g_n(x)}^{\nu((E_n)_x)}\,d\mu(x) \\
&= \int_X \nu(E_x)\,d\mu(x)
\end{aligned}\right\}\ (**)$$

Thus, E satisfies (ii).

This shows $\mathcal{C}$ is closed under countable increasing union.

It remains to show $\mathcal{C}$ is closed under countable decreasing intersections.

Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{C}$, $E_1 \supseteq E_2 \supseteq \ldots$

WTS $E := \bigcap_{n=1}^{\infty} E_n \in \mathcal{C}$.

Note that $(E_1)^y \supseteq (E_2)^y \supseteq \ldots$ $\forall x \in X$
$(E_1)_x \supseteq (E_1)_x \supseteq \ldots$ $y \in Y$.

Thus $f_n(y) := \mu(E_n^y)$ and $g_n(x) := \nu((E_n)_x)$ are nonneg, meas fns.

By cty from above, $\forall x \in X, y \in Y$, ← ($\mu(X), \nu(Y) < +\infty$)

$f_n(y) \searrow \mu(E^y)$, $g_n(x) \searrow \nu(E_x)$

Thus E satisfies (i).

Furthermore,

$0 \leq f_1(y) = \mu(E_1^y) \leq \mu(X) < +\infty$

$0 \leq g_1(x) = \nu((E_1)_x) \leq \nu(Y) < +\infty$

and bdd fns on finite measure spaces are integrable.

By practice midterm 2, Q2, (**) holds, so E satisfies property (ii). Hence $E \in \mathcal{C}$.

Thus $\mathcal{C}$ is a monotone class.

$\boxed{\text{Case 2}}$ Suppose μ, ν σ-finite

$\exists \{A_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$, $\bigcup_{i=1}^{\infty} A_i = X$, $\mu(A_i) < +\infty \ \forall i$

$\{B_i\}_{i=1}^{\infty} \subseteq \mathcal{N}$, $\bigcup_{i=1}^{\infty} B_i = Y$, $\nu(B_i) < +\infty \ \forall i$

WLOG, we may assume A_i and B_i are increasing. This ensures $\bigcup_{i=1}^{\infty} A_i \times B_i = X \times Y$.

Note that $\mu_i(A) := \mu(A \cap A_i)$
$\nu_i(B) := \nu(B \cap B_i)$
are finite measures.

Also, note that

$$\int 1_A \, d\mu_i = \int 1_A 1_{A_i} \, d\mu$$

$\Downarrow$ linearity

$$\int \varphi \, d\mu_i = \int \varphi 1_{A_i} \, d\mu \quad \forall \varphi \text{ simple}$$

$\Downarrow$ $\varphi_n \nearrow f$, monotone conv thm

$$\int f \, d\mu_i = \int f 1_{A_i} \, d\mu \quad \forall f \text{ nonneg, meas.}$$

By case 1, $\forall E \in \mathcal{M} \otimes \mathcal{N}$,

$x \mapsto \nu_i(E_x)$, $y \mapsto \mu_i(E^y)$ are meas

$$\int_X \nu_i(E_x) \, d\mu_i(x) = \int_Y \mu_i(E^y) \, d\nu_i(y).$$

By cty from below, $\forall x \in X, y \in Y$, $\nu_i(E_x) \nearrow \nu(E_x)$, $\mu_i(E^y) \nearrow \mu(E^y)$, so E satisfies (i) for μ and ν.

Also: $\nu_i(E_x)\mathbb{1}_{A_i}(x) \nearrow \nu(E_x)$ $(\star)$

$\mu_i(E^y)\mathbb{1}_{B_i}(y) \nearrow \mu(E^y)$.

By Monotone Convergence Thm,

$$\int_X \nu(E_x)\,d\mu(x)$$

$$\overset{(\star)}{=} \lim_{i\to\infty} \int_X \nu_i(E_x)\mathbb{1}_{A_i}(x)\,d\mu(x)$$

$$= \lim_{i\to\infty} \int_X \nu_i(E_x)\,d\mu_i(x)$$

μ_i, ν_i finite meas

$$= \lim_{i\to\infty} \int_Y \mu_i(E^y)\,d\nu_i(y)$$

$$= \ldots = \int_Y \mu(E^y)\,d\nu(y)$$

Thus E satisfies (ii) for μ and ν. $\square$