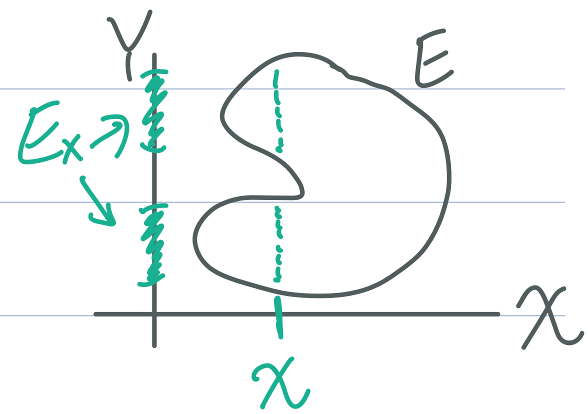
Lecture 16

ESCI's

Makeup lecture:

Tomorrow, Wed Nov 27th, 11-12:15

HSSB 4202



Recall:

Def: For any $E \in \mathcal{M} \otimes \mathcal{N}$, the <u>x-section</u> is $E_x = \{y : (x,y) \in E\}$

<u>y-section</u> $E^y = \{x : (x,y) \in E\}$

Basic properties of sections:

(A) If $E = A\times B$, $A\in\mathcal{M}$, $B\in\mathcal{N}$

$$E_x = \begin{cases} B & \text{if } x\in A \\ \emptyset & \text{if } x\notin A, \end{cases} \qquad E^y = \begin{cases} A & \text{if } y\in B \\ \emptyset & \text{if } y\notin B \end{cases}$$

(B) Given $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}\otimes\mathcal{N}$,

$$\left(\bigcup_{i=1}^{\infty} E_i\right)_x = \bigcup_{i=1}^{\infty} (E_i)_x$$

(C) Given $E\in\mathcal{M}\otimes\mathcal{N}$, $(E^c)_x = (E_x)^c$

Prop: If $E\in\mathcal{M}\otimes\mathcal{N}$ then

(*) $E_x\in\mathcal{N}$, $E^y\in\mathcal{M}$ $\forall x\in X, y\in Y$.

(D) If $E\in\mathcal{M}\otimes\mathcal{N}$, then

$$\nu(E_x) = \int_Y 1_{E_x}(y)\,d\nu(y) = \int_Y 1_E(x,y)\,d\nu(y)$$

Thm: Consider σ-finite measure spaces $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$.

For any $E \in \mathcal{M} \otimes \mathcal{N}$,

(i) the functions
$x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$
are $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ and $(\mathcal{N}, \mathcal{B}_{\mathbb{R}})$-meas.

(ii) $\int_X \nu(E_x)\,d\mu(x) = \int_Y \mu(E^y)\,d\nu(y)$

Spoiler: (ii) is equivalent to showing

$$\int_X \int_Y 1_E(x,y)\, d\nu(y)\, d\mu(x) = \int_Y \int_X 1_E(x,y)\, d\mu(x)\, d\nu(y)$$

$$=: \mu \otimes \nu(E)$$

we will define $\mu \otimes \nu$ in this way

<u>Lemma</u>: Consider measurable spaces $(X, \mathcal{M})$, $(Y, \mathcal{N})$. Let $\mathcal{A} := \{ \bigcup_{i=1}^{n} E_i : \{E_i\}_{i=1}^{n}$ are disjoint rectangles and $n \in \mathbb{N} \}$

Then $\mathcal{A}$ is an algebra and $\mathcal{M}(\mathcal{A}) = \mathcal{M} \otimes \mathcal{N}$.

Thm: Consider σ-finite measure spaces $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$.

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Define $\mu \otimes \nu : \mathcal{M} \otimes \mathcal{N} \to [0, +\infty]$ by

by previous theorem

$$\mu \otimes \nu(E) = \int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E^y)\, d\nu(y)$$

Then...

(i) $\mu \otimes \nu$ is a σ-finite measure on $\mathcal{M} \otimes \mathcal{N}$

(ii) $\mu \otimes \nu$ is the unique measure on $\mathcal{M} \otimes \mathcal{N}$ satisfying

$$\mu \otimes \nu(A \times B) = \mu(A)\, \nu(B) \quad \forall A \in \mathcal{M},\ B \in \mathcal{N}$$

with the convention $+\infty \cdot 0 = 0$

Pf: First, we show (i).

$$\mu\otimes\nu(\emptyset) = \int_X \underbrace{\nu(\overbrace{\emptyset_x}^{=\emptyset})}_{=0} d\mu(x) = 0$$

Next, if $\{E_i\}_{i=1}^\infty \subseteq \mathcal{M}\otimes\mathcal{N}$ disjoint, then, $\forall x\in X$, $\{(E_i)_x\}_{i=1}^\infty$ are disj.
Thus,

$$\mu\otimes\nu\left(\bigcup_{i=1}^\infty E_i\right) = \int_X \nu\Big(\overbrace{\Big(\bigcup_{i=1}^\infty E_i\Big)_x}^{=\bigcup_{i=1}^\infty (E_i)_x}\Big) d\mu(x)$$

$$= \int_X \sum_{i=1}^\infty \nu((E_i)_x)\, d\mu(x)$$

Beppo Levi

$$= \sum_{i=1}^\infty \int_X \nu((E_i)_x)\, d\mu(x)$$

$$= \sum_{i=1}^\infty \mu\otimes\nu(E_i)$$

Thus $\mu\otimes\nu$ is a measure.

Furthermore, for any rectangle $A\times B$, $A\in\mathfrak{M}$, $B\in\mathfrak{N}$,

$$\mu\otimes\nu(A\times B) = \int_X \overbrace{\nu((A\times B)_x)}^{1_A(x)\nu(B)}\, d\mu(x)$$
$$= \mu(A)\nu(B)$$

To see σ-finiteness, by σ-finiteness of $(X,\mathfrak{M},\mu)$, $(Y,\mathfrak{N},\nu)$,

WLOG increasing

$\exists \{A_i\}_{i=1}^{\infty} \subseteq \mathfrak{M}$, $\bigcup_{i=1}^{\infty} A_i = X$, $\mu(A_i) < +\infty\ \forall i$

$\exists \{B_i\}_{i=1}^{\infty} \subseteq \mathfrak{N}$, $\bigcup_{i=1}^{\infty} B_i = Y$, $\nu(B_i) < +\infty\ \forall i$

Thus $\bigcup_{i=1}^{\infty} A_i \times B_i = X\times Y$.

Finally,

$$\mu\otimes\nu(A_i\times B_i)=\mu(A_i)\nu(B_i)<+\infty \ \forall i.$$

It remains to show uniqueness. Suppose σ is another measure on $\mathcal{M}\times\mathcal{N}$ satisfying

$$\sigma(A\times B)=\mu(A)\nu(B),\ \forall A\in\mathcal{M}, B\in\mathcal{N}.$$

Let $\mathcal{A}$ be the algebra of finite disjoint unions of rectangles, so $\mathcal{M}(\mathcal{A})=\mathcal{M}\otimes\mathcal{N}$.

Note that, $\forall E\in\mathcal{A}$, we have

disjoint

$$E=\bigcup_{i=1}^{n} A_i\times B_i,\ \text{for some } \{A_i\}, \{B_i\},$$

and

$$\sigma(E) = \sum_{i=1}^{n} \sigma(A_i \times B_i) = \sum_{i=1}^{n} \mu(A_i)\nu(B_i)$$

$$\ldots = \sum_{i=1}^{n} \mu\otimes\nu(A_i \times B_i) = \mu\otimes\nu(E).$$

Furthermore, we just showed $\mu\otimes\nu$ satisfies strong σ-finiteness hypothesis on $\mathcal{A}$.

Thus by Thm on uniqueness of measures,

$$\sigma(E) = \mu\otimes\nu(E) \quad \forall\, E \in \mathcal{M}(\mathcal{A}) = \mathfrak{M}\otimes\mathfrak{N}$$

$\square$

MAJOR THM 15

Thm: Consider σ-finite measure spaces $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$.

Tonelli: Given $f: X \times Y \to [0, +\infty]$ $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable, then

(i) $x \mapsto \int_Y f(x,y)\, d\nu(y)$ is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-meas

(ii) $y \mapsto \int_X f(x,y)\, d\mu(x)$ is $(\mathcal{N}, \mathcal{B}_{\overline{\mathbb{R}}})$-meas

(iii) $$\int_{X \times Y} f(x,y)\, d\mu \otimes \nu(x,y) = \int_X \left[\int_Y f(x,y)\, d\nu(y) \right] d\mu(x)$$
$$= \int_Y \left[\int_X f(x,y)\, d\mu(x) \right] d\nu(y)$$

(*)

Fubini: Given $f \in L^1(\mu \otimes \nu)$

(a) $y \mapsto f(x,y)$ is in $L^1(\nu)$ for μ-a.e. x

(b) $x \mapsto f(x,y)$ is in $L^1(\mu)$ for ν-a.e. y

only defined for μ-a.e. x

(c) $x \mapsto \int_Y f(x,y)\,d\nu(y)$ is in $L^1(\mu)$

(d) $y \mapsto \int_X f(x,y)\,d\mu(x)$ is in $L^1(\nu)$

(e) $(*)$ holds

Rmk: Often to verify

$$\int |f|\,d\mu\otimes\nu < +\infty$$

in order to apply Fubini, one first uses Tonelli.

Pf:

For any $E \in \mathcal{M} \times \mathcal{N}$, we have already shown that $x \mapsto \nu(E_x)$, $y \mapsto \mu(E^y)$ are meas and

$$\int_Y 1_E(x,y)\,d\nu(y) \qquad \int_X \left[\int_Y 1_E(x,y)\,d\nu(y)\right] d\mu(x)$$

$$\int_{X \times Y} 1_E \, d\mu \otimes \nu = \int_X \nu(E_x)\,d\mu(x)$$

$$= \int_Y \mu(E^y)\,d\nu(y)$$

$$\int_Y \left[\int_X 1_E(x,y)\,d\mu(x)\right] d\nu(y)$$

Thus, Tonelli holds for $f = 1_E$

By linearity of the integral, we see the theorem holds for simple function.

Suppose $f \geq 0$ and $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}_{\overline{\mathbb{R}}})$-meas. $\exists\, \varphi_n$ simple s.t. $\varphi_n \nearrow f$ pointwise.

Thus...

- $\int \varphi_n(x,y)\,d\nu(y) \overset{\text{MCT}}{\nearrow} \int f(x,y)\,d\nu(y)$ $\forall x \in X$, so (i) holds.
- interchanging ν & μ, (ii) holds.
- $$\int f\,d\mu\otimes\nu \overset{\text{MCT}}{=} \lim_{n\to\infty} \int \varphi_n\,d\mu\otimes\nu$$
  $$= \lim_{n\to\infty} \int\int \varphi_n\,d\mu\,d\nu$$
  $$\overset{\text{MCT}}{=} \int\left[\lim_{n\to\infty}\int \varphi_n\,d\mu\right]d\nu$$
  $$\overset{\text{MCT}}{=} \int\int \underbrace{\lim_{n\to\infty} \varphi_n}_{f}\,d\mu\,d\nu$$

and likewise interchanging μ + ν, so (iii) holds.

This completes the proof of Tonelli.

It remains to show Fubini. Since $f \in \mathcal{L}^1(\mu \otimes \nu)$,

$$+\infty > \int |f| \, d\mu \otimes \nu = \iint |f| \, d\mu \, d\nu = \int \left(\int |f| \, d\nu \right) d\mu$$

so $\int |f(x,y)| \, d\nu(y) < +\infty$ μ-a.e.-x $\Rightarrow$ (a)
$\int |f(x,y)| \, d\mu(x) < +\infty$ ν-a.e.-y $\Rightarrow$ (b)

Consequently, for μ-a.e. x,

$$\left|\int f(x,y)\,d\nu(y)\right| \le \int |f(x,y)|\,d\nu(y)$$

Hence,

$$\int\left|\int f(x,y)\,d\nu(y)\right|d\mu(x) \le \iint |f(x,y)|\,d\nu(y)\,d\mu(x) \overset{<+\infty}{}$$

This shows (c), and (d) similar.

Finally, to show (e), since $x \mapsto f(x,y)$ is $L^1(\mu)$ for ν-a.e y,

$$\iint f(x,y)\,d\mu(x)\,d\nu(y)$$

$$=\int\left[\int (f(x,y))_+\,d\mu(x) - \int (f(x,y))_-\,d\mu(x)\right]d\nu(y)$$

$$\ldots = \int\int (f(x,y))_+ \, d\mu(x)\, d\nu(y) \qquad \text{linearity of integral OK by } \star$$

$$- \int\int (f(x,y))_- \, d\mu(x)\, d\nu(y)$$

Tonelli

$$= \int\int (f(x,y))_+ \, d\nu(y)\, d\mu(x)$$

$$- \int\int (f(x,y))_- \, d\nu(y)\, d\mu(x)$$

linearity of integral

$$= \int\int f(x,y)\, d\nu(y)\, d\mu(x).$$

Finally, since $f \in L^1(\mu \otimes \nu)$

$$\int f \, d\mu\otimes\nu = \int (f)_+ \, d\mu\otimes\nu$$

$$- \int (f)_- \, d\mu\otimes\nu \qquad \text{Tonelli}$$

$$= \int\int (f)_+ \, d\mu\, d\nu$$

$$- \int\int (f)_- \, d\mu\, d\nu$$

Combining with above shows (e).