Lecture 17

Thm: Consider σ-finite measure spaces $(X, \mathfrak{M}, \mu)$, $(Y, \mathfrak{N}, \nu)$.

MAJOR THEOREM 14

Define $\mu \otimes \nu : \mathfrak{M} \otimes \mathfrak{N} \to [0, +\infty]$ by

by previous theorem

$$\mu \otimes \nu (E) = \int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E^y)\, d\nu(y)$$

Then...

(i) $\mu \otimes \nu$ is a σ-finite measure on $\mathfrak{M} \otimes \mathfrak{N}$

(ii) $\mu \otimes \nu$ is the unique measure on $\mathfrak{M} \otimes \mathfrak{N}$ satisfying

$$\mu \otimes \nu (A \times B) = \mu(A)\, \nu(B) \quad \forall A \in \mathfrak{M}, B \in \mathfrak{N}$$

with the convention $+\infty \cdot 0 = 0$

MAJOR THM 15

Thm: Consider σ-finite measure spaces $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$.

Tonelli: Given $f: X \times Y \to [0, +\infty]$ $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable, then

(i) $x \mapsto \int_Y f(x,y)\, d\nu(y)$ is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-meas

(ii) $y \mapsto \int_X f(x,y)\, d\mu(x)$ is $(\mathcal{N}, \mathcal{B}_{\overline{\mathbb{R}}})$-meas

(iii) $$\int_{X \times Y} f(x,y)\, d\mu\otimes\nu(x,y) = \int_X \left[\int_Y f(x,y)\, d\nu(y) \right] d\mu(x)$$
$$= \int_Y \left[\int_X f(x,y)\, d\mu(x) \right] d\nu(y)$$

(*)

Fubini: Given $f \in L^1(\mu \otimes \nu)$

(a) $y \mapsto f(x,y)$ is in $L^1(\nu)$ for μ-a.e. x

(b) $x \mapsto f(x,y)$ is in $L^1(\mu)$ for ν-a.e. y

only defined for μ-a.e. x

(c) $x \mapsto \int_Y f(x,y)\,d\nu(y)$ is in $L^1(\mu)$

(d) $y \mapsto \int_X f(x,y)\,d\mu(x)$ is in $L^1(\nu)$

(e) $(*)$ holds

Rmk: Often to verify

$$\int |f|\,d\mu\otimes\nu < +\infty$$

in order to apply Fubini, one first uses Tonelli.

Rmk: It may seem "obvious" that one should be able to interchange the order of integration, but this can be very powerful, just like interchanging limit and integral.

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d-dimensional measures, $d > 2$

How do we take the product of $\{(X_i, \mathcal{M}_i, \mu_i)\}_{i=1}^{d}$ σ-finite?

product σ-algebras:

By practice midterm (proved, for simplicity, for $d=3$), we have

$$\bigotimes_{i=1}^{d} \mathcal{M}_i = (((\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3)) \otimes \mathcal{M}_4 \ldots$$

by symmetry of LHS wrt indexing

$$= \ldots \otimes (\mathcal{M}_{d-2} \otimes (\mathcal{M}_{d-1} \otimes \mathcal{M}_d))$$

$=$ any order of taking pairwise products

<u>product measures:</u> By HW7,

$((\mu_1 \otimes \mu_2) \otimes \mu_3) \otimes \ldots$

$=$ any order of taking pairwise products

This motivates how can define product of d measures, $d \geq 2$.

Cor (product of d measures): Given $\{(X_i, \mathcal{M}_i, \mu_i)\}_{i=1}^{d}$ σ-finite,

$$\bigotimes_{i=1}^{d} \mu_i := ((\mu_1 \otimes \mu_2) \otimes \mu_3) \otimes \dots$$

↖ or any other choice of pairwise products

is the unique measure on $\bigotimes_{i=1}^{d} \mathcal{M}_i$, s.t. for any rectangle

$$R := \prod_{i=1}^{d} R_i, \quad R_i \in \mathcal{M}_i,$$

$$\left(\bigotimes_{i=1}^{d} \mu_i\right)(R) = \prod_{i=1}^{d} \mu_i(R_i).$$

Pf: This follows by applying theorem on uniqueness of pairwise products of measures.

The following theorem shows that our construction of the product measure coincides with that appearing in many other texts.

Thm: Given $\{(X_i, \mathcal{M}_i, \mu_i)\}_{i=1}^d$ σ-finite, for any $E \in \bigotimes_{i=1}^d \mathcal{M}_i$, let $\mu := \bigotimes_{i=1}^d \mu_i$.

$$\mu(E) = \inf\left\{\sum_{j=1}^\infty \mu(R_j) : R_j \text{ rect and } E \subseteq \bigcup_{j=1}^\infty R_j\right\}$$

$$\mu(R_j) = \prod_{i=1}^d \mu_i(R_{j,i}), \quad R_j = \prod_{i=1}^d R_{j,i}.$$

Pf: Define $\nu(E) := \text{RHS}$.

Note that, if $E \subseteq \bigcup_{j=1}^\infty R_j$, R_j rect, then subadditivity ensures

$\mu(E) \le \sum_{j=1} \mu(R_j)$.

Hence $\mu(E)$ is a lower bound for the set. Thus $\mu(E) \le \nu(E)$.

Furthermore, if R is a rectangle $\mu(R)$ is an element of the set, so $\nu(R) \le \mu(R) \Rightarrow \nu(R) = \mu(R)$.

Since μ is the unique measure on $\bigotimes_{i=1}^{d} \mathcal{M}_i$ that gives the right size to rectangles, it remains to show that ν is a measure on $\bigotimes_{i=1}^{d} \mathcal{M}_i$.

By HW3, Q4, ν is an outer measure on $\prod_{i=1}^{d} X_i$.

It remains to show

$$\bigotimes_{i=1} \mathcal{M}_i \subseteq \underbrace{\mathcal{M}_\nu}_{\text{Carathéodory } \sigma\text{-algebra}},$$

which holds as long as all rectangles are ν-measurable.

Fix $E \subseteq \prod_{i=1}^{d} X_i$, rectangle R.

WLOG $\nu(E) < +\infty$. Fix $\varepsilon > 0$ arb.

There exists $\{R_j\}_{j=1}^{\infty}$ rect, $E \subseteq \bigcup_{j=1}^{\infty} R_j$ s.t.

$$\underbrace{\nu(E) + \varepsilon}_{\text{NOT a lower bound of the set}} \geq \sum_{j=1}^{\infty} \mu(R_j)$$

μ is a measure, hence additive

$$= \sum_{j=1}^{\infty} \mu(\underbrace{R_j \cap R}_{= \bigcup_k R_{j,k}}) + \mu(\underbrace{R_j \cap R^c}_{\bigcup_\ell R_{j,\ell}})$$

by the fact that the set of finite disjoint unions of rectangles is an algebra!

disjoint

finite sum for each j

$$= \dots = \sum_{j=1}^{\infty} \sum_{k} \mu(R_{j,k}) + \sum_{\ell} \mu(R_{j,\ell})$$

$$\geq \nu(E \cap R) + \nu(E \cap R^c)$$

Since $E \subseteq \bigcup_{j=1}^{\infty} R_j \Rightarrow E \cap R \subseteq \bigcup_{j=1}^{\infty} R_j \cap R = \bigcup_{j=1}^{\infty} \bigcup_{k} R_{j,k}$,

and likewise for $E \cap R^c$.

Since $\varepsilon > 0$ was arb, we conclude $R \in \mathcal{M}_\nu$. $\square$

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d-dimensional Lebesgue measure

While it *is* conventional to consider $\bigotimes_{i=1} \mu_i$ on $\bigotimes_{i=1} \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^d}$

for any collection of Lebesgue Stieltjes measures μ_j ...

... we do not define d-dim'l Lebesgue measure λ^d by $\bigotimes_{i=1}^{d} \lambda$ on $\bigotimes_{i=1}^{d} \mathcal{M}_{\lambda^*}$.

Why? It's not complete (for $d>1$). (HW7)

<u>Recall</u>: HW3, Q9: completion of measure measure space $(X, \mathcal{M}, \mu)$.

$\mathcal{N} := \{ N \in \mathcal{M} : \mu(N) = 0 \}$ (null sets)

Def: Given a measure space $(X, \mathcal{M}, \mu)$, if $\mathcal{M}$ includes all subsets of null sets, then μ is complete.

$$\forall F \subseteq N \in \mathcal{N},\ F \in \mathcal{M}.$$

Folland Thm 1.9: The completion of a σ-algebra $\mathfrak{M}$

$$\overline{\mathfrak{M}} := \{E \cup F : E \in \mathfrak{M},\ F \subseteq N \in \mathcal{N}\}$$

is a σ-algebra and $\exists!$ extension of μ to $\overline{\mathfrak{M}}$ given by

$$\bar{\mu}(E \cup F) := \mu(E).$$

HW3, Q9: $(\lambda, \mathfrak{M}_{\lambda^*})$ is completion of $(\lambda, \mathcal{B}_{\mathbb{R}})$.

Def (d-dimensional Lebesgue measure)
λ^d is the completion of

$$\left(\bigotimes_{i=1}^{d} \lambda, \mathcal{B}_{\mathbb{R}^d}\right).$$

Let $\mathcal{M}_{\lambda^d} := \overline{\mathcal{B}_{\mathbb{R}^d}}$.

Thm: For all $A \in \mathcal{M}_{\lambda^d}$,

$$\lambda^d(A) = \inf\left\{\sum_{j=1}^{d} \lambda^d(R_j) : A \subseteq \bigcup_{j=1}^{\infty} R_j, R_j \text{ rect}\right\}$$

Pf:

$\boxed{\leq}$ $A \subseteq \bigcup_{j=1}^{\infty} R_j \Rightarrow \lambda^d(A) \leq \sum_{j=1}^{\infty} \lambda^d(R_j)$

Thus $\lambda^d(A)$ is a lower bound for the set.

$\boxed{\geq}$ WLOG, assume $\lambda^d(A) < +\infty$.
By defn of λ^d, $\mathcal{B}_{\mathbb{R}^d} \subseteq$

$A = E \cup F$, for $E \in \mathcal{B}_{\mathbb{R}^d}$, $F \subseteq N \in \mathcal{N}$
WLOG E, F disjoint otherwise $F := F \setminus E$.
Thus $\lambda^d(E) < +\infty$.

Fix $\varepsilon > 0$. Applying previous theorem to E and N,
$\exists \{R_j^E\}_{j=1}^\infty$, $\{R_j^N\}_{j=1}^\infty$ s.t.
$E \subseteq \bigcup_{j=1}^\infty R_j^E$, $F \subseteq N \subseteq \bigcup_{j=1}^\infty R_j^N$ and

$$\lambda^d(E) + \frac{\varepsilon}{2} \geq \sum_{j=1}^\infty \lambda^d(R_j^E), \quad \underbrace{\lambda^d(N)}_{0=} + \frac{\varepsilon}{2} \geq \sum_{j=1}^\infty \lambda^d(R_j^N)$$

Thus,

$$\lambda^d(A) + \varepsilon = \lambda^d(E) + \underbrace{\lambda^d(F)}_{\lambda^d(F) \leq \lambda^d(N) = 0} + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$\geq \sum_{j=1}^\infty \lambda^d(R_j^E) + \sum_{j=1}^\infty \lambda^d(R_j^N).$$

Since $A = E \cup F \subseteq \left(\bigcup_{j=1}^{\infty} R_j^E\right) \cup \left(\bigcup_{j=1}^{\infty} R_j^N\right)$,

$\lambda^d(A) + \varepsilon$ is <u>not</u> a lower bound for the set.

Therefore, we conclude $\lambda^d(A)$ is the greatest lower bound for the set. $\square$

<u>Rmk</u>: HW7, Q3: $\bigotimes_{i=1}^{d} \mathfrak{M}_{\lambda^*} \subsetneq \mathfrak{M}_{\lambda^d}$.

<u>Happy Thanksgiving!</u>