Lecture 1

Why measure theory?

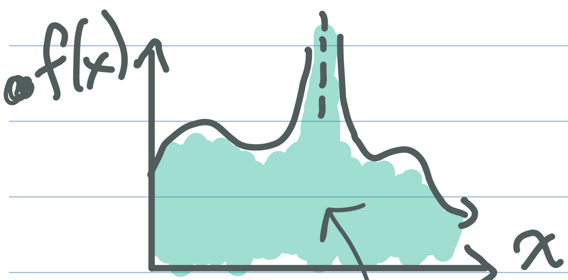
Q: What is the "total length" of an arbitrary set $E \subseteq \mathbb{R}$?

Can we define a function $\mu : \overbrace{2^{\mathbb{R}^d}}^{\text{power set}} \to [0, +\infty]$
so $\mu(E) =$ "d-dimensional measure of E"

Properties we'd like to have...

- For $d = 1$, $E = [a,b]$, want $\mu(E) = b - a$.
- For $\{I_i\}_{i=1}^n$ disjoint intervals, we'd want

$$\mu\left(\bigcup_{i=1}^n I_i\right) = \sum_{i=1}^n \mu(I_i)$$

- But what about $E = \mathbb{Q} \cap [0,1]$?
- 

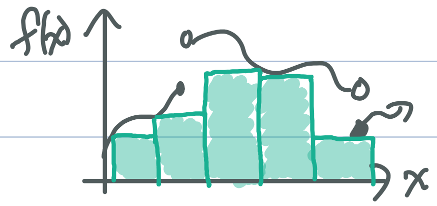
  What about the are a under the graph of $f(x)$, for $f(x)$ arbitrary?

$$\int_{\mathbb{R}} f(x)\,dx$$

(mid 1800s)

Pre-measure theory: Riemann Integral

- rigorous defn of integral of, e.g., p.w. cts fns in terms of upper/lower sums



- $\boxed{\text{PROS}}$ good enough for "ordinary" functions
- $\boxed{\text{CONS}}$ not good for taking limits

For example, given $f_1, f_2, f_3, \ldots : [a,b] \to \mathbb{R}$
s.t. $\underbrace{\lim_{n\to\infty} f_n(x) = f(x) \quad \forall\, x \in [a,b]}_{\text{"pointwise convergence"}}$.

when can we conclude that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx \, ?$$

Measure theory: a much more powerful theory of integration than the Riemann integral

$\boxed{\text{PROS}}$:
- a much larger class of functions is Lebesgue integrable
- better for taking limits of fns
- consequently better suited for probability, functional analysis, PDE

$\boxed{\text{CONS}}$: None :)

First goal: define a function $\mu : 2^{\mathbb{R}} \to [0,+\infty)$ satisfying the following

① If $\{E_i\}_{i=1}^{n} \subseteq 2^{\mathbb{R}}$ (or $\{E_i\}_{i=1}^{\infty} \subseteq 2^{\mathbb{R}}$) are disjoint, then

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i) \quad \left(\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)\right)$$

Def: If $\mu: 2^{\mathbb{R}} \to [0, +\infty]$ satisfies criteria ①, it is <u>finitely additive</u> (<u>countably additive</u>).

② $\mu([a,b]) = b - a$

③ $\mu(\underbrace{E+c}_{=\{x+c \,:\, x \in E\}}) = \mu(E)$ for all $c \in \mathbb{R}$, $E \subseteq \mathbb{R}$

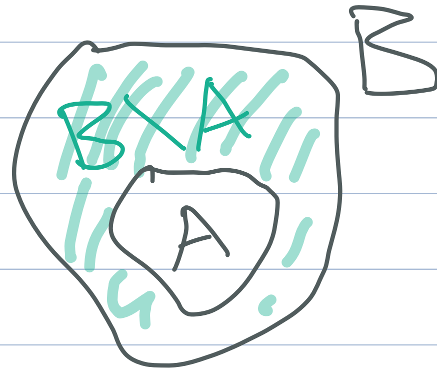
Def: If $\mu: 2^{\mathbb{R}} \to [0, +\infty]$ satisfies criteria ③, it is <u>translation invariant</u>.

<u>**Thm**</u>: (Vitali) No such function exists.

<u>**Lemma**</u>: (monotonicity) Given a set X and $\mu: 2^X \to [0, +\infty)$ is finitely additive, then $\forall A, B \subseteq X$,

$$A \subseteq B \Rightarrow \mu(A) \leq \mu(B).$$

Pf: Finite additivity and nonnegativity imply



$$\mu(B) = \mu(A \cup (B \setminus A))$$
$$= \mu(A) + \mu(B \setminus A)$$
$$\geq \mu(A) \qquad \square$$

<u>Pf of Thm:</u>

Assume, for the sake of contradiction that such a μ exist.

Define an equivalence relation on $\mathbb{R}$:

$$x \sim y \iff x - y \in \mathbb{Q}.$$

$$[x] := \{y \in \mathbb{R} : y \sim x\}$$

<u>Claim 1</u>: Every equivalence class contains an ~~e~~ element in $[0,1]$.

For each equivalence class, choose an element in $[0,1]$ belonging to that class, and call the resulting set A.

Warning: Axiom of Choice
Bogachev 1.12(x)

Let $B = \bigcup_{q \in \mathbb{Q} \cap [-1,1]} A + q$

Claim 2: this is a disjoint union
Pf: HW1

Claim 3: $[0,1] \overset{(i)}{\subseteq} B \overset{(ii)}{\subseteq} [-1,2]$

(i) For any $x \in [0,1]$, $x \in [a]$ for some $a \in A$. Thus, $x - a = q$ for $q \in \mathbb{Q}$.
Since $x \in [0,1]$, $a \in [0,1]$, $q \in [-1,1]$.
Thus $x \in B$.

(ii)

If $b \in B$, then $b = a + q$ for some $a \in A \subseteq [0,1]$ and $q \in [-1,1]$, thus $b \in [-1,2]$.

By monotonicity lemma and criterion ②,

$$1 = \mu([0,1]) \leq \mu(B) \leq \mu([-1,2]) = 3$$

However, by criteria ① and ③,

$$\mu(B) = \sum_{q \in \mathbb{Q} \cap [-1,1]} \mu(A+q) = \sum_{q \in \mathbb{Q} \cap [-1,1]} \mu(A)$$

Since $\mu(B) \leq 3$, $\mu(A) = 0$.

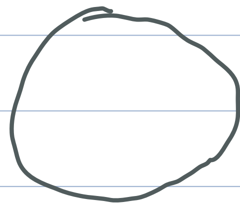
Thus $\mu(B) = 0$.

This contradicts that $\mu(B) \geq 1$. $\square$

Which criterion do we weaken to get existence of such a ~~measure~~ notion of size?

If we weaken ① to finite additivity, there are still problems for $d \geq 3$:

Banach-Tzarski Paradox (1924)

$U = \bigcup_{i=1}^{n} E_i$  $V = \bigcup_{i=1}^{n} F_i$

F_i is a rotation/translation of E_i.

Bogachev 1.12(xi)

If we weaken criteria ② or ③, no longer compatible with usual notion of length.

Two good choices:

measure

- don't require μ to be defined on all of $2^{\mathbb{R}}$.
- still define μ on all of $2^{\mathbb{R}}$, but replace countable additivity with countable subadditivity.

that is, for any $\{E_i\}_{i=1}^{\infty}$ disjoint,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

outer measure

We'll see that...

- given a measure on a "nice" family of subsets of $\mathbb{R}$, it extends to an outer measure on $2^{\mathbb{R}}$
- given an outer measure, we can single out "nice sets" on which it is a measure.

Measures

What kind of family of subsets should we restrict to?

Let X be a set.

Def: $\mathcal{A} \subseteq 2^X$ is an <u>algebra of subsets of X</u> if it is nonempty and

① $E_1, \dots, E_n \in \mathcal{A} \Rightarrow \boxed{\bigcup_{i=1}^{n} E_i \in \mathcal{A}}$

② $E \in \mathcal{A} \Rightarrow \boxed{E^c \in \mathcal{A}}$

"closed under finite unions"

"closed under complements"