Lecture 2

Announcements: Office Hours, Mid 2 Date

Recall:

<u>Def</u>: $\mu: 2^{\mathbb{R}^d} \to [0,+\infty]$ is <u>finitely additive</u> (resp. <u>countably additive</u>) if, for $\{E_i\}_{i=1}^n \subseteq 2^{\mathbb{R}^d}$ (resp. $\{E_i\}_{i=1}^\infty \subseteq 2^{\mathbb{R}^d}$) disjoint, we have $\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i)$ (resp. $\mu\left(\bigcup_{i=1}^\infty E_i\right) = \sum_{i=1}^\infty \mu(E_i)$).

<u>Def</u>: $\mu: 2^{\mathbb{R}^d} \to [0,+\infty]$ is <u>translation invariant</u> if $\mu(E+c) = \mu(E)$ for all $E \in 2^{\mathbb{R}^d}$, $c \in \mathbb{R}^d$.

Thm (Vitali): There is no function $\mu: 2^{\mathbb{R}} \to [0,+\infty]$ that is countably additive, translation invariant and satisfies $\mu([a,b]) = b-a \quad \forall\ a \leq b$.

First way to fix this problem: restrict μ to a family of "nice" sets.

Let X be a nonempty set.

Def: $\mathcal{A} \subseteq 2^X$ is an algebra of subsets of X if it is nonempty and

① $E_1, \dots, E_n \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{n} E_i \in \mathcal{A}$

② $E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A}$

"closed under finite unions"

"closed under complements"

Lemma: If $\mathcal{A}$ is an algebra, then

① $\emptyset \in \mathcal{A}$, $X \in \mathcal{A}$

② $E_1, \dots, E_n \in \mathcal{A} \Rightarrow \bigcap_{i=1}^{n} E_i \in \mathcal{A}$

"closed under finite intersections"

Pf: HW2

Ex:

(i) $\mathcal{A} = \{\emptyset, X\}$

(ii) $\mathcal{A} = 2^X$

(iii) Let $\mathcal{A}$ be the collection of all finite and cofinite subsets of X.

Def: $\mathcal{A} \subseteq 2^X$ is a <u>σ-algebra</u> <u>of subsets of X</u> if it is an algebra and if it is closed under countable unions.

Remark: A σ-algebra is also closed under countable intersections.

Remark: Any algebra $\mathcal{A}$ that is closed under <u>countable disjoint unions</u>, that is,

$$\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A} \text{ disjoint} \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$$

is a σ-algebra.

Ex: (i) and (ii) are σ-algebras. (iii) isn't always.

Def: Given a nonempty set X and a σ-algebra $\mathcal{M} \subseteq 2^X$, we will call $(X, \mathcal{M})$ a measurable space. We will call $E \in \mathcal{M}$ a measurable set.

Prop: Given any $\mathcal{E} \subseteq 2^X$, there exists a $\boxed{\text{smallest}}$ σ-algebra containing $\mathcal{E}$, which we denote $\mathcal{M}(\mathcal{E})$ and refer to as the σ-algebra generated by $\mathcal{E}$.

That is, all other σ-algebras $\mathcal{F}_1$ that contain $\mathcal{E}$ satisfy $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{F}_1$.

Pf:

Claim: Given any nonempty collection $\mathcal{C}$ of σ-algebras on X, then

$$\bigcap \mathcal{C} := \{E \subseteq X : E \in \mathcal{A} \ \forall \mathcal{A} \in \mathcal{C}\}$$

is a σ-algebra.

Pf: HW2

Let $\mathcal{C} = \{\mathcal{A} : \mathcal{A} \subseteq 2^X \text{ is a } \sigma\text{-alg}, \mathcal{E} \subseteq \mathcal{A}\}$. Since $2^X \in \mathcal{C}$, we see that $\mathcal{C}$ is nonempty.

By CLAIM, $\bigcap \mathcal{C}$ is a σ-algebra.

By defn of $\mathcal{C}$, $\mathcal{E} \subseteq \bigcap \mathcal{C}$ and for any σ-algebra $\mathcal{A}$ s.t. $\mathcal{E} \subseteq \mathcal{A}$, $\bigcap \mathcal{C} \subseteq \mathcal{A}$.

Thus $\mathcal{M}(\mathcal{E}) = \bigcap \mathcal{C}$. $\square$

<u>Rmk</u>: Intuitively, $\mathcal{M}(\mathcal{E})$ creates a σ-algebra containing $\mathcal{E}$ by "going from the outside in," that is, with σ-algebras that are too big and taking intersections.

Recall: a topology $\mathcal{T}$ on X is a collection of subsets of X that is closed under arbitrary unions and finite intersections. elements of $\mathcal{T}$ are "open sets"

Let $(X, \mathcal{T})$ be a topological space.

Def: The Borel σ-algebra of X, denote $\mathcal{B}(X)$, is the σ-algebra generated by $\mathcal{T}$. Its members are known as Borel sets.

What do the Borel sets "look like"? Go from inside out...

<u>Notation</u>: $\mathcal{F} \subseteq 2^X$

$\mathcal{F}^\sigma$:= all countable unions of elements of $\mathcal{F}$

$\mathcal{F}^\delta$:= all countable inters. of elements of $\mathcal{F}$,

$\overline{\mathcal{F}}$:= all complements of elements of $\mathcal{F}$

To build $\mathcal{B}(X)$ from inside out...

$$\mathcal{T} \to \mathcal{T}^\delta \to \mathcal{T}^\delta \cup \left(\overline{\mathcal{T}^\delta}\right) \to \underbrace{\ldots}_{\text{uncountably many steps}} \to \mathcal{B}(X)$$

"Borel hierarchy"

Prop: The Borel σ-algebra of $\mathbb{R}$, denote $\mathcal{B}_{\mathbb{R}}$, is generated by each of the following

(i) open intervals, $\mathcal{E}_1 = \{(a,b) : a < b\}$

(ii) closed " ", $\mathcal{E}_2 = \{[a,b] : a \leq b\}$

(iii) half-open " ", $\mathcal{E}_3 = \{[a,b) : a < b\}$

(iv) open rays, $\mathcal{E}_4 := \{(a,+\infty) : a \in \mathbb{R}\}$

(v) closed rays, $\mathcal{E}_5 := \{[a,+\infty) : a \in \mathbb{R}\}$

Pf: HW 2

Measures

Def: Given a measurable space $(X, \mathcal{M})$, a measure is a function $\mu : \mathcal{M} \to [0,+\infty]$ s.t.

(i) $\mu(\emptyset) = 0$

(ii) given $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ disjoint,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

We call $(X, \mathcal{M}, \mu)$ a <u>measure space</u>.

<u>Ex</u>: (Dirac mass / Dirac measure)

$(X, \mathcal{M}) = (X, 2^X)$

Fix $x_0 \in X$ and define

$$\mu(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{otherwise} \end{cases}$$

Often denoted $\mu = \delta_{x_0}$.

Ex: (counting measure)

$(X, \mathcal{M}) = (X, 2^X)$

$\mu(A) = \#$ of elements in A

Thm: For any measure space $(X, \mathcal{M}, \mu)$ and $A, B \in \mathcal{M}$, $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$,

(i) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$

(ii) $A \subseteq B$, $\mu(A) < +\infty$

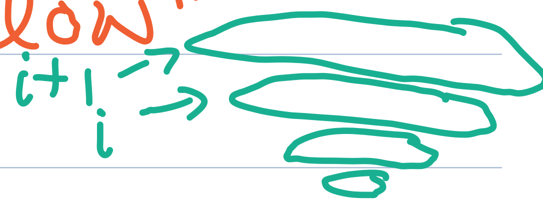
$\Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A)$

not necessarily disjoint

(iii) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$

(iv) $A_i \subseteq A_{i+1} \; \forall i \Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu(A_i)$

"continuity from below"



(v) $A_{i+1} \subseteq A_i \ \forall i$, $\mu(A_1) < +\infty$
$\Rightarrow \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i)$

"continuity from above"

Pf:

(i) See Lec 1

(ii) $\mu(B) = \mu((B \setminus A) \cup A) = \mu(B \setminus A) + \mu(A)$

Since $\mu(A) < +\infty$, we may subtract $\mu(A)$ to obtain result.

(iii) Define $B_1 = A_1$, $B_2 = A_2 \setminus A_1, \ldots,$
$B_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$.

Then $\{B_n\}_{n=1}^{\infty}$ is disjoint and $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$.

Thus,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$$

$B_n \subseteq A_n$, by part (i)

$$\leq \sum_{n=1}^{\infty} \mu(A_n).$$

(iv) next time ◡̈